National Physical Laboratory

A Beginner’s Guide to Convolution and Deconvolution

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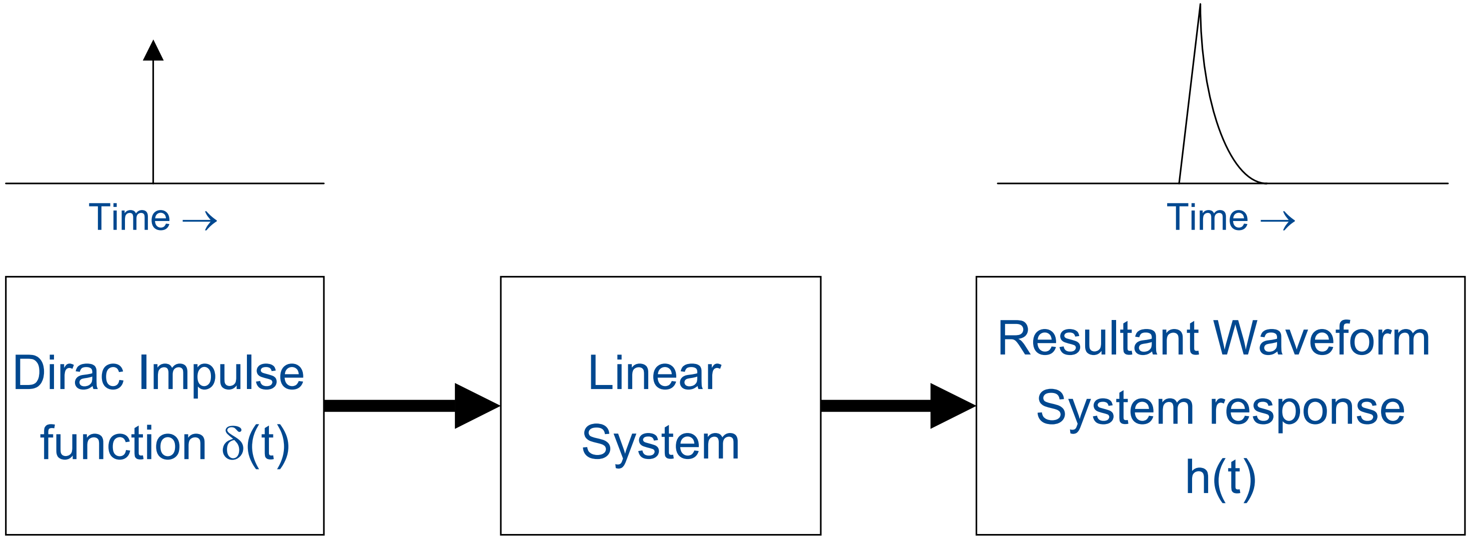
Overview

- Introduction
- Pre-requisites
- Convolution and correlation
- Fourier transform deconvolution
- Direct deconvolution
- Summary

Convolution

- Finite impulse response for a system
- Convolution implies history/memory of the stimulus
- Convolution implies Bandwidth



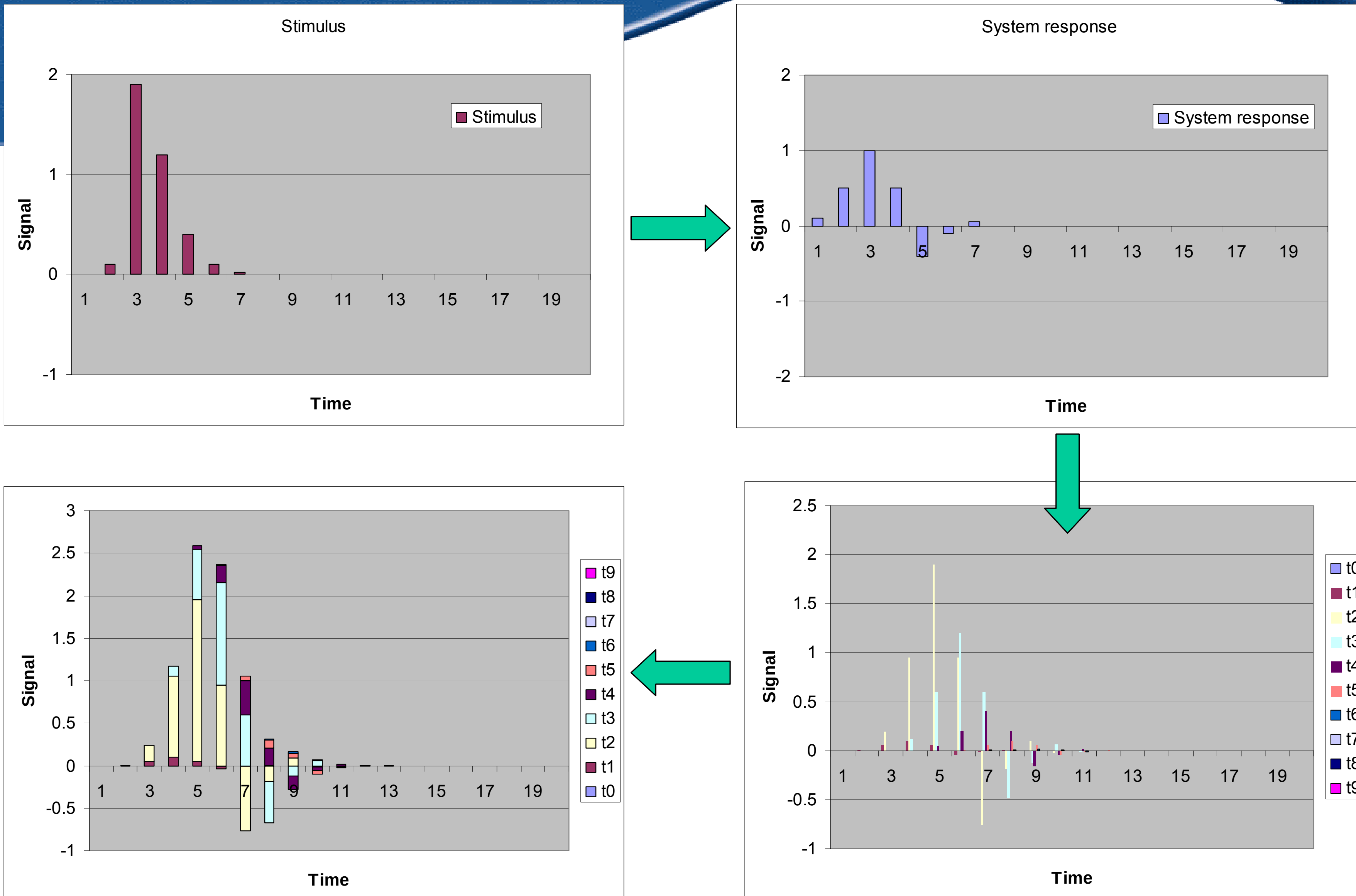
Prerequisites

- System must be linear
- $f(a)+f(b)=f(a+b)$
- Superposition must apply
- V_{out} = Gain x V_{in} Linear system
- V_{out} = A x $V1_{in}$ x $V2_{in}$ Nonlinear system
- Linearise
- Keep $V2_{in}$ constant

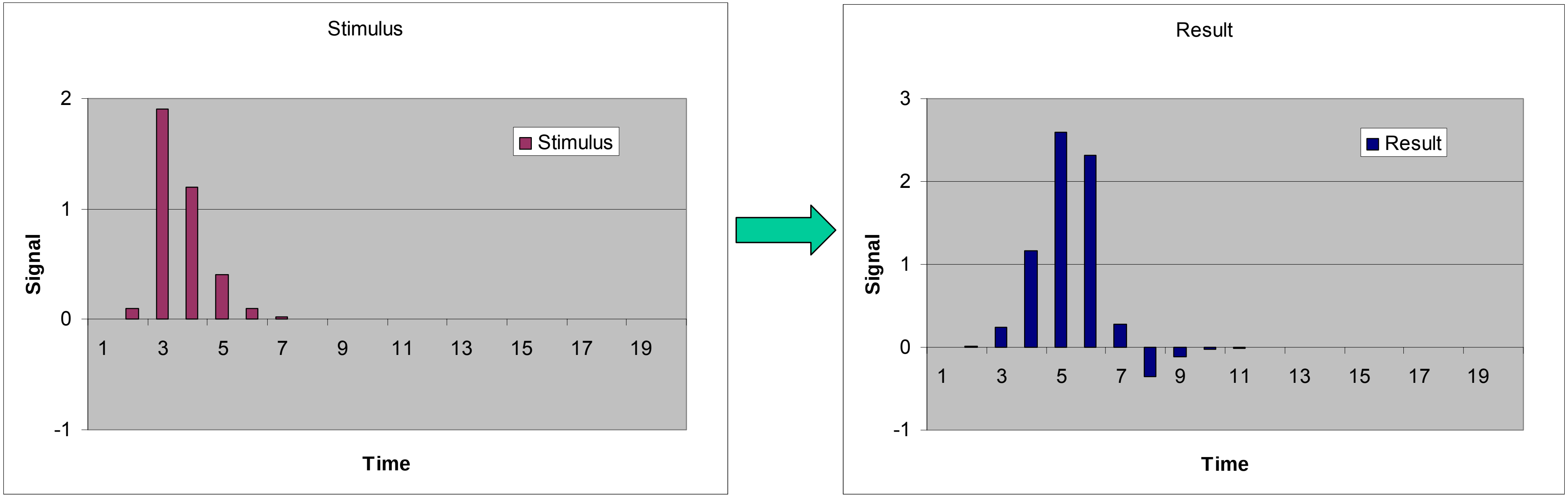
Prerequisites

- System can be described as a waveform – impulse response
- Waveform must be time invariant
- Shape of f(t) unchanged through translation $f(t-t_1)$

Convolution

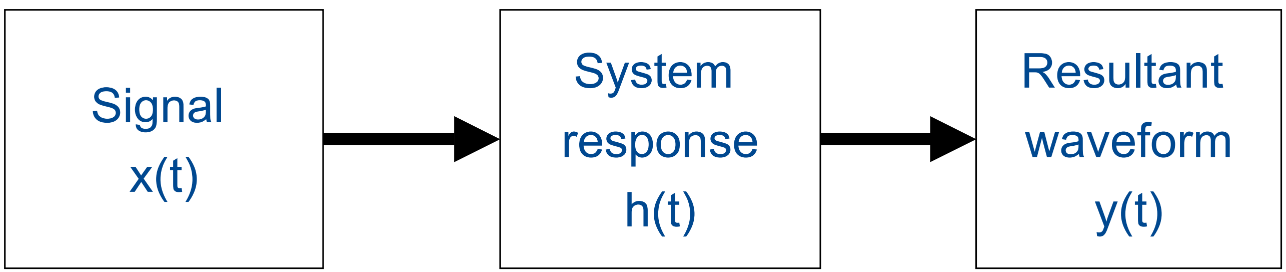


Convolution



Measured waveform

- Convolution of stimulus and system response



Frequency domain

$$Y(s) = H(s)X(s)$$

Time domain

$$y(t) = \int_{-\infty}^{+t} x(\tau)h(t-\tau)d\tau$$

$$y_i = \sum_{j=0}^{i} x_j h_{i-j}$$

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Convolution and Correlation

- Correlation – the direction of signal is reversed

Frequency domain

Time domain

Convolution

$$Y(s) = H(s)X(s)$$

$$y(t) = \int_{-\infty}^{+t} x(\tau)h(t-\tau)d\tau$$

Correlation

$$Y(s) = H(s)\overline{X}(s)$$

$$y(t) = \int_{-\infty}^{+t} x(\tau)h(t+\tau)d\tau$$

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Application areas

- Optics and Image capture
- Waveform capture/processing
- Electronic Engineering
- Communications
- Acoustics

De-convolution

Deconvolution

- Estimating the underlying signal from the smoothed result
- Convolution with an inverse filter

- Convolution rules apply (Linearity, Superposition, Time invariance)

- ILL-POSED PROBLEM – may not have a perfect solution

Applications

- Image analysis and correction
- Instrument response correction
- Waveform analysis
- Oscilloscope risetime calibration
- Transducer response calibration
- Geological surveying
- Echo cancelling/line frequency response correction (Modem/Broadband ADSL)

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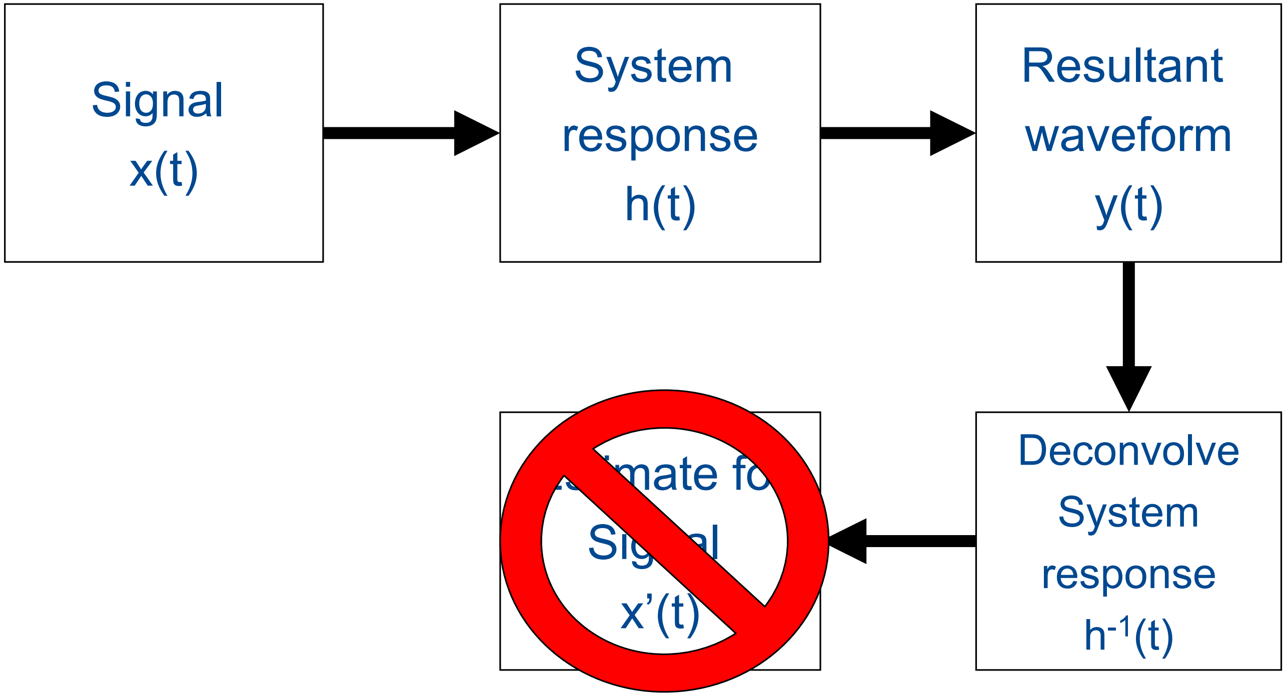
Deconvolution of measured waveform

- Convolution of stimulus and system response
- Deconvolution – correction for the system response



$h^{-1}(t)$ is the inverse of the system response h(t)

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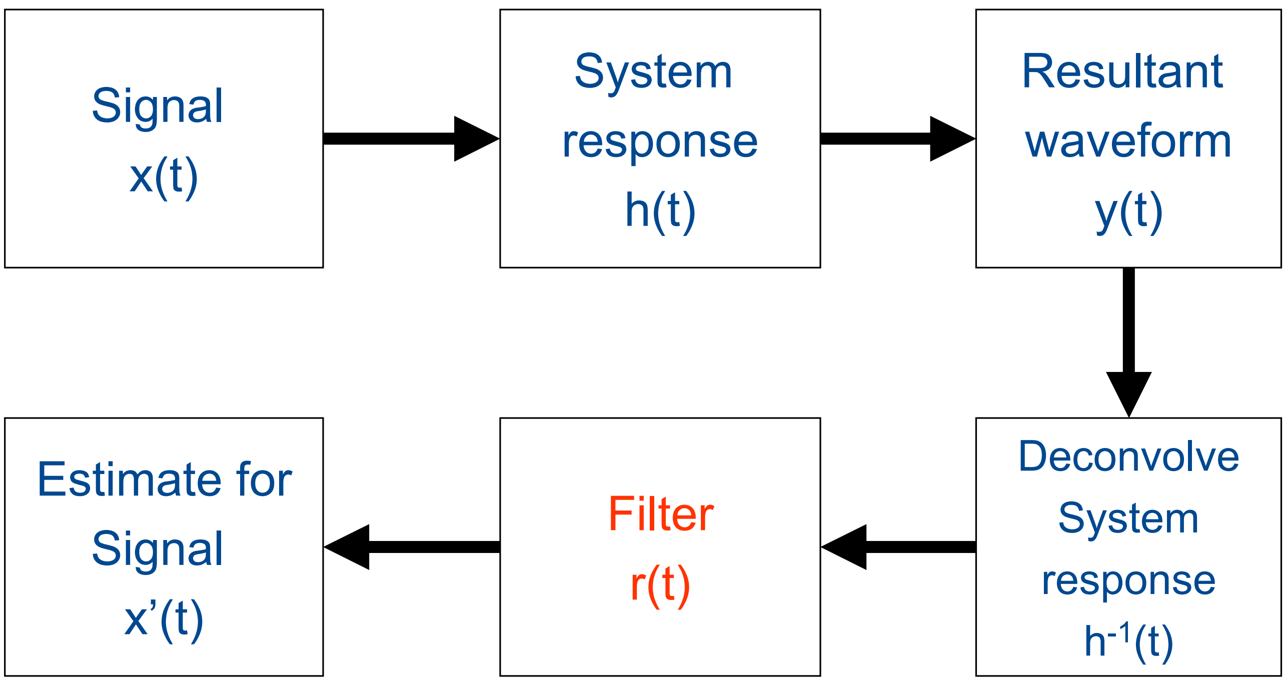
Deconvolution of measured waveform

- Convolution of stimulus and system response
- Deconvolution – correction for the system response



$h^{-1}(t)$ is the inverse of the system response h(t)

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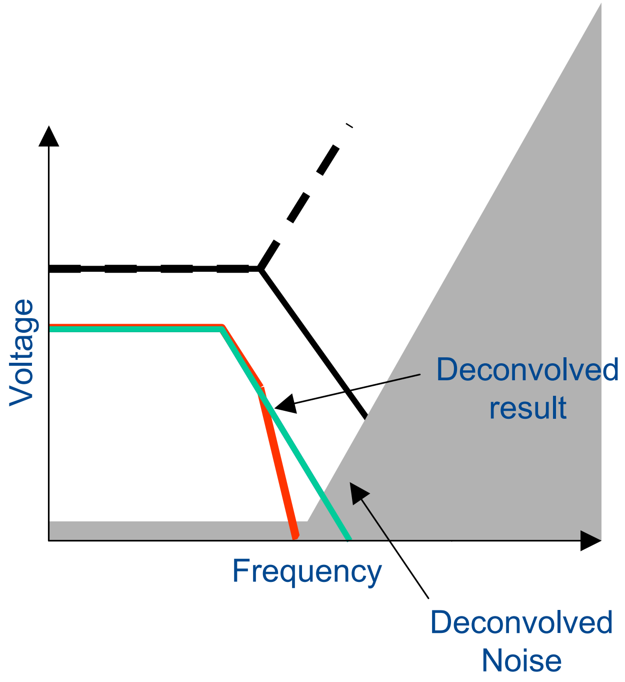
Deconvolution

$$X(j\omega) = \frac{Y(j\omega) + noise}{H(j\omega)}$$

- Inverse problem
- Ill-posed
- Noise
- System response errors



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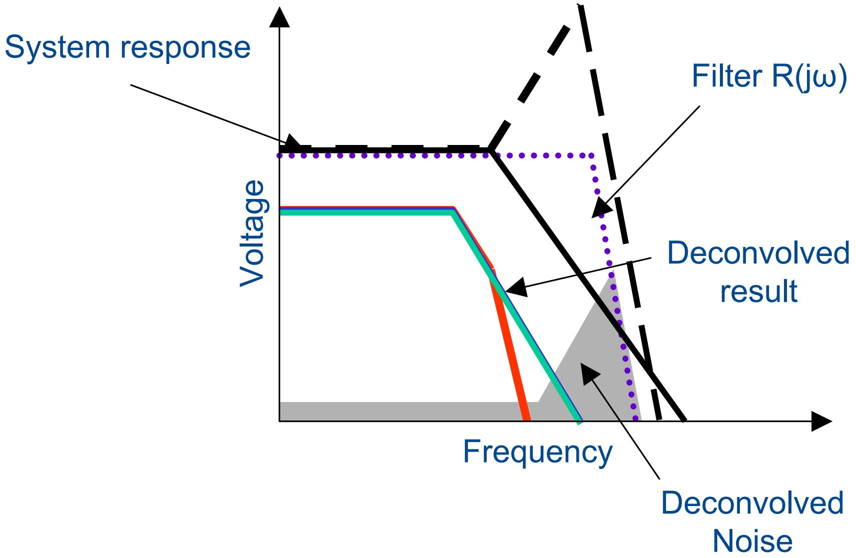
Deconvolution with filter

$$X(j\omega) = \frac{Y(j\omega) + noise}{H(j\omega)} R(j\omega)$$

- Inverse problem
- Ill-posed
- Noise
- System response errors
- Filter added to limit noise/errors



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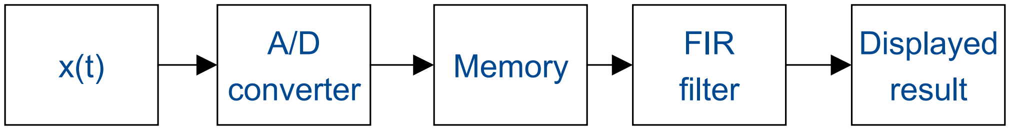
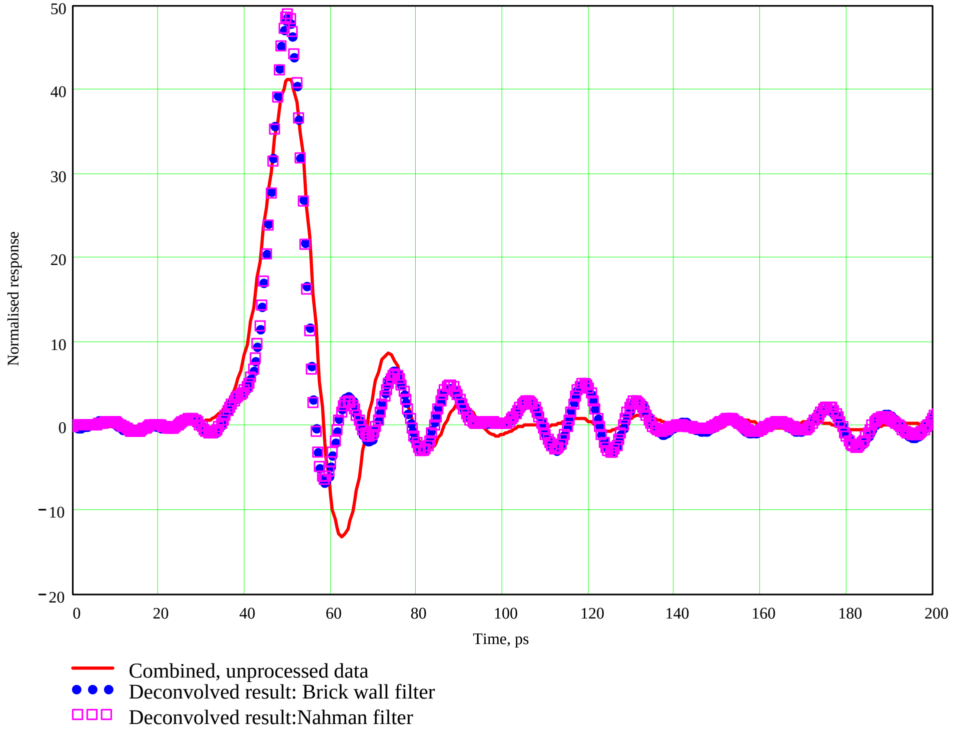
Realisation of Deconvolution process



- System impulse response
- Direct time-domain convolution (Digital filtering - FIR)
- Transform approach (e.g. Fourier transform)

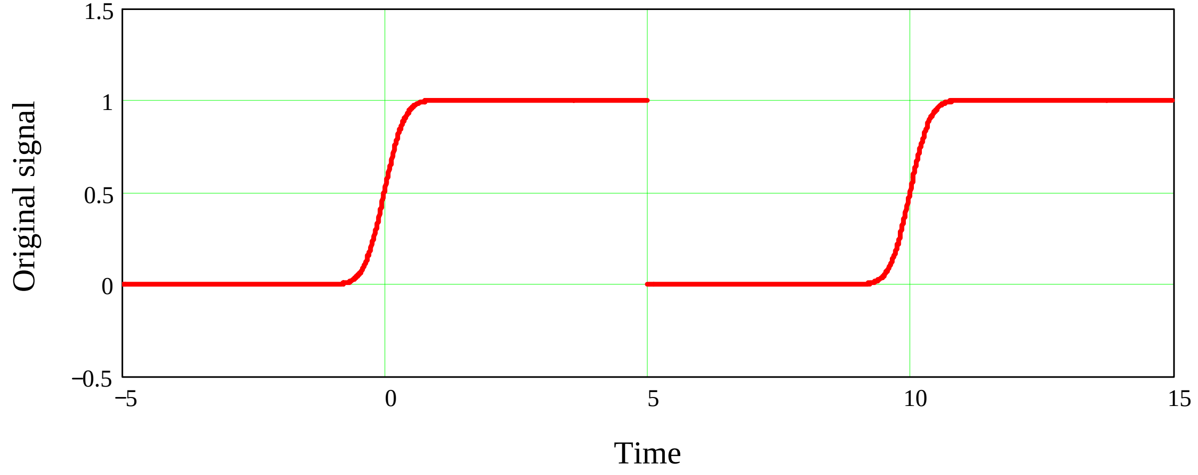
$$R(j\omega) = \frac{|H(j\omega)|^2}{|H(j\omega)|^2 + \alpha |C|^2}$$

1. α is a user controlled parameter
2. C may be constant or frequency dependent e.g. ω^2 maximises the smoothness of the result



Limitations

- Fourier Transform assumes a cyclic repeating signal
- Fourier Transform deconvolution cannot be applied to step waveforms directly



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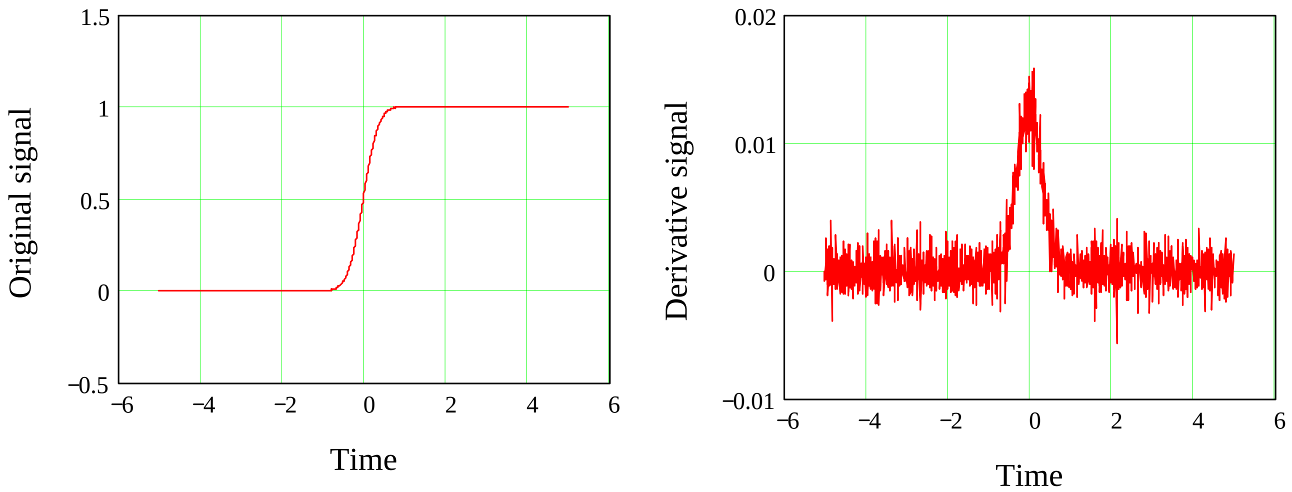
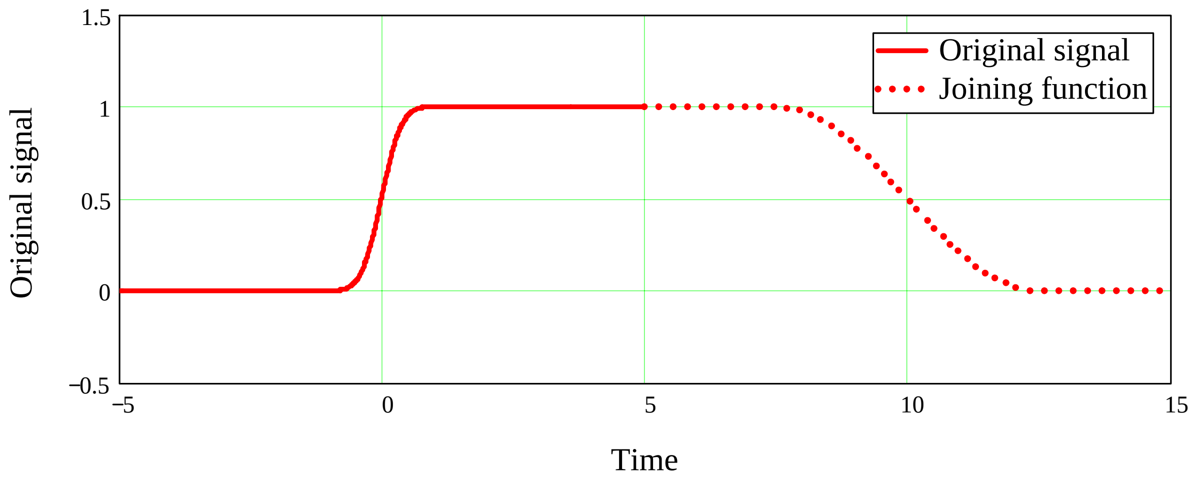
Solution 1

- Differentiate the signal $x_i \rightarrow x_i - x_{i-1}$
- Deconvolve the signal
- Integrate the signal $x_i \rightarrow x_i + x_{i-1}$



Solution 2

- Append a ‘joining function’ e.g. A cos(πt/T)+B to the signal
- Deconvolve the signal
- Discard the joining function



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Direct deconvolution

- Convolution described as a summation
- Deconvolution as a least-squares solution for x?
- Filtering required to smooth result

Convolution sum

$$y_i = \sum_{j=0}^{i} x_j h_{i-j}$$

Least-squares representation – minimise errors

$$\left(\sum_{j=0}^{i} x'_j \, h_{i-j} - y_i \right)^2 = 0$$

Additional least-squares equations – maximise smoothness of the result

$$\gamma \left(x'_{j-1} - 2x'_{j+1} + x'_{j+1} \right)^2 = 0$$

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Direct deconvolution

- Large system of equations (>1000 unknowns)
- Direct deconvolution of step response is possible

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