Manual of Codes of Practice for the Determination of Uncertainties in Mechanical Tests on Metallic Materials

Code of Practice No. 13

The Determination of Uncertainties in Dynamic Young's Modulus

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CONTENTS

1. SCOPE

This procedure covers the evaluation of uncertainty in the determination of dynamic Young's modulus and related quantities in elastic materials at ambient temperatures according to the testing Standard:

ASTM E 1876 –97[1], *"Dynamic Young's Modulus, Shear Modulus, and Poisson's Ratio by Impulse Excitation of Vibration",* American Society for Testing and Materials, 1997.

ASTM E1876-97 describes how the resonant frequencies of elastic materials are excited by striking a rectangular or cylindrical bar which is free to vibrate. A transducer and associated electronic equipment measure the frequency which can be related through a knowledge of the bar's dimensions and mass, and the material's Poisson's ratio, to the dynamic Young's modulus. In the case of the rectangular bar, the fundamental flexural frequency can be excited and similarly used to calculate the dynamic shear modulus. Knowledge of both the dynamic Young's modulus and shear modulus can be used to determine Poisson's ratio where this is otherwise unknown.

2. SYMBOLS AND DEFINITIONS

For a complete list of symbols and definitions of terms on uncertainties, see Section 2 of the main Manual [1]. It should be noted that not all the symbols and definitions of terms on uncertainties used in this CoP are consistent with the GUM [2].

The following list gives the symbols and definitions used in this procedure.

A empirical correction factor for dynamic shear modulus dependent on the width to cross-section ratio of a rectangular bar
b width of the bar
B empirical correction factor for dynamic shear modulus dependent on the width to cross-section ratio of a rectangular bar
c_i sensitivity coefficient
COD coefficient of determination
CoP Code of Practice
D diameter of a cylindrical bar
d_v divisor used to calculate the standard uncertainty
E dynamic Young's modulus
F axial force
f_f fundamental resonant frequency of a rectangular or cylindrical bar in flexure
f_t fundamental resonant frequency of a bar in torsion

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[1] This ASTM standard is commonly used in Europe as no suitable alternative ISO, EN, or national standard exists.

G	dynamic shear modulus
k	coverage factor used to calculate expanded uncertainty (usually corresponding to 95% confidence level) where a normal probability distribution **can** be assumed
k_p	coverage factor used to calculate expanded uncertainty (usually corresponding to 95% confidence level) where a normal probability distribution **cannot** be assumed
L	length of the bar
m	mass of the bar
n	number of repeat measurements
μ	Poisson's ratio
p	confidence level
q	random variable
$\bar{q}$	arithmetic mean of the values of the random variable x
s	experimental standard deviation (of a random variable) determined from a limited number of measurements, n
T	nominal test temperature (in degrees Celsius or Kelvin, as indicated)
t	thickness of the bar
T_1	correction factor for dynamic Young's modulus, for fundamental flexure mode of a rectangular bar
$T_1{}'$	correction factor for dynamic Young's modulus, for fundamental flexure mode of a cylindrical bar
u	standard uncertainty
u_c	combined standard uncertainty
U	expanded uncertainty
x_i	estimate of input quantity
y	test (or measurement) result
v_i	degrees of freedom of standard uncertainty u
v_{eff}	effective degrees of freedom used to obtain k_p

3.   INTRODUCTION

There are requirements for test laboratories to evaluate and report the uncertainty associated with their test results. Such requirements may be demanded by a customer who wishes to know the bounds within which the reported result may be reasonably assumed to lie; or the laboratory itself may wish to understand which aspects of the test procedure have the greatest effect on results so that this may be monitored more closely or improved. This Code of Practice has been prepared within UNCERT, a project partially funded by the European Commission's Standards, Measurement and Testing programme under reference SMT4 -CT97-2165 to simplify the way in which uncertainties in mechanical test on metallic materials are evaluated. The aim is to avoid ambiguity and provide a common format readily understandable by customers, test laboratories and accreditation authorities.

This Code of Practice is one of seventeen prepared and tested by the UNCERT consortium for the estimation of uncertainties in mechanical tests on metallic materials. The Codes of Practice have been collated in a single Manual [1] which has the following sections:

1. Introduction to the evaluation of uncertainty.
2. Glossary of definitions and symbols
3. Typical sources of uncertainty in materials testing
4. Guidelines for the estimation of uncertainty for a test series.
5. Guidelines for reporting uncertainty.
6. Individual Codes of Practice (of which this is one) for the estimation of uncertainties in mechanical tests on metallic materials.

This CoP can be used as a stand-alone document. Nevertheless, for background information on measurement uncertainty and values of standard uncertainties of devices used commonly in material testing, the user may need to refer to the relevant section in Reference 1. Several sources of uncertainty, such as the reported tolerance of load cells, extensometers, micrometers and thermocouples, are common to several mechanical tests and are included in Section 3 of the Manual [1]. These are not discussed here to avoid needless repetition. The individual procedures are kept as straightforward as possible by following the same structure:

- The main procedure.
- Fundamental aspects and major contributions to the uncertainty for the specific test type (Appendix A)
- A worked example (Appendix B)

This document guides the user through several steps to be carried out in order to estimate the uncertainty in dynamic Young's modulus. The general process for calculating uncertainty values is described in section 1 of the Manual [1].

4. A PROCEDURE FOR ESTIMATING THE UNCERTAINTY IN DYNAMIC YOUNG'S MODULUS

Step 1. Identifying of the Parameters for which Uncertainty is to be Estimated

The first step is to list the quantities (measurands) for which the uncertainties must be calculated. Table 1 shows the parameters that usually constitute the results of a test performed to ASTM E 1876-97. Generally, dynamic Young's modulus, dynamic shear modulus and Poisson's ratio are considered to be the primary results of this testing standard. These measurands are not measured directly, instead they are determined from other quantities (or measurements).

Table 1. Measurands, Measurements, their Units and Symbols

Measurands	Units	Symbol
Dynamic Young's modulus	Pa	*E*
Dynamic shear modulus	Pa	*G*
Poisson's ratio	dimensionless	μ

Measurements	Units	Symbol
Mass of the bar	Kg	*m*
Width of the bar	m	*b*
Length of the bar	m	*L*
Thickness of the bar	m	*t*
Fundamental resonant frequency of a rectangular bar, or circular rod, in flexure	Hz	f_f
Fundamental resonant frequency of rectangular bar in torsion	Hz	f_t
Diameter of rod	m	*D*
Test temperature	°C	T

Step 2. Identifying all Sources of Uncertainty in the Test

In Step 2, the user must identify all possible sources of uncertainty that may have an effect (directly or indirectly) on the test. This list cannot be exhaustively identified beforehand as it is uniquely linked to the test procedure and the apparatus used. This means that, a new list should be drafted each time a particular test parameter changes (when a plotter is replaced by a computer for example). To help the user list all sources of uncertainty, four categories have been defined. Table 2 lists the four categories and some examples of sources in each category.

It is important to note that Table 2 is NOT exhaustive and is for GUIDANCE only - relative contributions may vary according to the material tested and the test conditions. Individual laboratories are encouraged to prepare their own list to correspond to their own test facility, and assess the associated significance of the contributions.

To simplify the uncertainty calculations it may be advisable to regroup the significant sources of uncertainty in Table 2 according to the following categories:

1   Uncertainty associated with the test piece measurements
2   Uncertainty in associated with the apparatus
3   Uncertainty due to the environment
4   Uncertainty due to operator or procedure

Appendix A presents the fundamental aspects of the above categorisation.

Table 2. Typical Sources of Uncertainty and Their Likely Contribution to Uncertainties on Dynamic Young's Modulus at Ambient Temperature

[1 = major contribution, 2 = minor contribution, blank = insignificant (or no) contribution]

Source of uncertainty	Type [1]	Measurand		
		E	*m*	*G*
1. Test Piece				
Micrometer / calliper / operator errors in measuring bar dimensions	A or B	1	1	1
Shape tolerance, edge effects	B	2	2	2
Accuracy of balance	A or B	2	2	2
Not isotropic	B		2	
2. Apparatus				
Damping from supports	B	2	2	2
Damping from transducer	B	2	2	2
Accuracy of transducer/ electronics	B	1	1	1
3. Environment				
Poor control of ambient temperature	B	2	2	2
Humidity - effect on test piece moisture content	B	2	2	2
4. Method				
Incorrect calculation of T_1	B	1	2	
Omission of A	B		2	1
Incomplete iterative solution of Poisson's ratio	B		2	

Step 3. Classifying Sources of Uncertainty according to Type A or B.

In accordance with ISO TAG4 *'Guide to the Expression of Uncertainties in Measurement'*, the sources of uncertainty can be classified as Type A or B, depending on how their influence is quantified. If the uncertainty is evaluated by statistical means (from a number of repeated observations), it is classified Type A. If it is evaluated by any other means it should be classified as Type B.

The values associated with Type B uncertainties can be obtained from a number of sources including calibration certificates, manufacturer's information, or an expert's estimation. For Type B sources, it is necessary for the user to estimate for each source the most appropriate probability distribution (further details are given in Section 2 of Reference 1).

Attention should be drawn to the fact that one source can be classified as either Type A or B depending on how it is estimated. For example, if the width of a rectangular test bar is measured once, that uncertainty associated with repeatability of measurements is considered Type B. If the mean value of two or more consecutive measurements is taken into account, then the uncertainty is Type A.

Table 3 shows   worksheets containing typical sources of uncertainty and their type for dynamic young's modulus, shear modulus and Poisson's ratio.

Table 3a. Typical Worksheet for Uncertainty Budget Calculations for Estimating the Uncertainty in Dynamic Young's Modulus of a Rectangular Bar.

Source of uncertainty	Symbol	Value	Probability distribution	Divisor d_v	c_i	$u_i(E)$ Pa	v_i or v_{eff}
Mass of the bar	m		rectangular	$\sqrt{3}$	1		∞
Width of the bar	b		rectangular	$\sqrt{3}$	1		∞
Length of the bar	L		rectangular	$\sqrt{3}$	3		∞
Thickness of the bar	t		rectangular	$\sqrt{3}$	3		∞
Fundamental frequency in flexure	f_f		rectangular	$\sqrt{3}$	2		∞
Mean value of E	$u(E)_{rep}$		normal	1	1		n-1
Combined standard uncertainty	u_c		normal			$u_c(E)$	$\mathbf{n_{eff}}$
Expanded uncertainty	U		normal				$\mathbf{n_{eff}}$

1) See Equation A15
2) The incomplete iterative solution of ***m*** will have a computable but negligible effect on *E* through Equation A3.

Table 3b. Typical Worksheet for Uncertainty Budget Calculations for Estimating the Uncertainty in Shear Modulus of a Rectangular Bar.

Source of uncertainty	Symbol	Value	Probability distribution	Divisor d_v	c_i	$u_i(G)$ Pa	v_i or v_{eff}
Mass of the bar	m		rectangular	$\sqrt{3}$	1		∞
Width of the bar	b		rectangular	$\sqrt{3}$	1		∞
Length of the bar	L		rectangular	$\sqrt{3}$	1		∞
Thickness of the bar	t		rectangular	$\sqrt{3}$	1		∞
Fundamental frequency in torsion	f_t		rectangular	$\sqrt{3}$	2		∞
Mean value of G	$u(G)_{rep}$		normal	1	1		n-1
Combined standard uncertainty	U_c		normal			$u_c(G)$	$\mathbf{n_{eff}}$
Expanded uncertainty	U		normal				$\mathbf{n_{eff}}$

1) See Equation A16
2) The omission of the correction factor A is for most practical purposes a bias, resulting in an overestimate of *G*.

Table 3c. Typical Worksheet for Uncertainty Budget Calculations for Estimating the Uncertainty in Poisson's Ratio of a Rectangular Bar

Source of uncertainty	Symbol	Value	Probability distribution	Divisor d_v	c_I	$u_i(m)$	v_i or v_{eff}
Dynamic Young's modulus	E		normal	1	1		∞
Dynamic Shear modulus	G		normal	1	1		∞
Mean value of **m**	$u(m)_{rep}$		normal	1	1		n-1
Combined standard uncertainty	U_c		normal			$u_c(m)$	n_{eff}
Expanded uncertainty	U		normal				n_{eff}

see Equation A17

Step 4. Estimating the Standard Uncertainty and Sensitivity Coefficient for each Source of Uncertainty.

In Step 4, the standard uncertainty, u, for each input source identified in Table 2 is estimated (see Appendix A). The standard uncertainty is defined as one standard deviation and is derived from the uncertainty of the input quantity divided by the parameter, d_v, associated with the assumed probability distribution. The divisors for the distributions most likely to be encountered are given in Section 2 of Ref. [1].

The standard uncertainty requires the determination of the associated sensitivity coefficient, c, which is usually estimated from the partial derivatives of the functional relationship between the output quantity (the measurand) and the input quantities. The calculations required to obtain the sensitivity coefficients by partial differentiation can be a lengthy process, particularly when there are many individual contributions and uncertainty estimates are needed for a range of values. If the functional relationship for a particular measurement is not known, the sensitivity coefficients may be obtained experimentally. In many cases the input quantity to the measurement may not be in the same units as the output quantity. For example, one contribution to dynamic Young's modulus, E, is the rectangular bar's width, b. In this case the input quantity, b, is measured in metres, but the output quantity, E, is in Pascals. In such case a sensitivity coefficient, c_i (corresponding to the partial derivative of the relationship between E and b) is used to convert from the width to the dynamic Young's modulus (for more information see Appendix A).

The calculation of the sensitivity coefficients and review of the relative contributions of each source of uncertainty (Appendix A) leads to the following general conclusions for measurements of a rectangular bar:

- Most of the uncertainty in dynamic Young's modulus is contributed by the uncertainty in measuring the fundamental frequency in flexure. Secondary contributions are made by

measuring the thickness of the bar, and its length (particularly if calipers are used for the latter).

- The uncertainty in dynamic Shear modulus in a rectangular bar is dominated by the uncertainty in measuring the fundamental frequency in torsion. Other contributions are generally negligible.

- The uncertainty in Poisson's ratio is formed in almost equal proportions by uncertainties in dynamic Young's modulus, and by dynamic Shear modulus. Hence the major contributing sources are the measurement of frequencies.

To help in the process of calculation, it is useful to summarise the uncertainty analysis in a spreadsheet - or 'uncertainty budget'- as in Tables 3a, 3b and 3c. Appendix A includes the mathematical formulae for calculating the uncertainty contributions and Appendix B contains a worked example.

Step 5. Computing the Combined Uncertainty u_c.

Assuming that individual uncertainty sources are uncorrelated, the measurand's combined uncertainty, $u_c(y)$, can be computed using the root sum squares:

$$u_c(y) = \sqrt{\sum_{i=1}^{N} [c_i . u(x_i)]^2} \qquad (1)$$

where c_i is the sensitivity coefficient associated with measurement x_i. This uncertainty corresponds to plus or minus one standard deviation on the normal distribution law representing the studied quantity. The combined uncertainty has an associated confidence level of 68.27%.

This calculation of the combined uncertainty is also included in Table 3, for each of the measurands, *E, G* and ***m*** Uncertainties in E and G, contribute to the combined uncertainty in μ, since μ is wholly dependent on those two measurands.

Step 6. Computing the Expanded Uncertainty U.

The expanded uncertainty, U, is defined in Reference 2 as "the interval about the result of a measurement that may be expected to encompass a large fraction of the distribution of values that could **reasonably** be attributed to the measurand". It is obtained by multiplying the combined uncertainty, u_c calculated in step 5, by a coverage factor, k, which is selected on the basis of the level of confidence required. For a normal probability distribution, the most generally used coverage factor is 2 which corresponds to a confidence interval of 95.4% (effectively 95% for most practical purposes). The expanded uncertainty, U, is, therefore, broader than the combined uncertainty, u_c. Where a high confidence level is demanded by the customer (such as for

aerospace industry, electronics), a coverage factor of 3 is often used so that the corresponding confidence level increases to 99.73%.

In cases where the probability distribution of u_c is not normal (or where the number of data points use in a Type A analysis is small), the value of k should be calculated from the degrees of freedom given by the Welsh-Satterthwaite method (see Reference 1, Section 4 for more details).

Step 7. Reporting results.

Once the expanded uncertainty has been estimated, the results should be reported in the following way:

$$V = y \pm U \qquad (2)$$

where V is the estimated value of the measurand, y is the test (or measurement) result, U is the expanded uncertainty associated with y. An explanatory note, such as that given in the following example should be added (change as appropriate):

"The reported expanded uncertainty is based on a standard uncertainty multiplied by a coverage factor, k=2, which for a normal distribution corresponds to a coverage probability, p, of approximately 95%. The uncertainty evaluation was carried out in accordance with UNCERT CoP 13: 2000."

5.   REFERENCES

1.   *Manual of Codes of Practice for the determination of uncertainties in mechanical tests on metallic materials.* Project UNCERT, EU Contract SMT4-CT97-2165, Standards Measurement & Testing Programme, ISBN 0-946754-41-1, Issue 1, September 2000.

2.   BIPM, IEC, IFCC, ISO, IUPAC, OIML, "Guide to the expression of uncertainty in measurement". International Organisation for Standardisation, Geneva, Switzerland, ISBN 92-67-10188-9, First Edition, 1993. [This Guide is often referred to as the GUM or the ISO TAG4 document after the ISO Technical Advisory Group that drafted it.]

   Identical documents:
   - CEN, ENV 13005.
   - BSI, *"Vocabulary of metrology, Part 3. Guide to the expression of uncertainty in measurement"*, PD 6461: Part 3 : 1995, British Standards Institution.

3.   *"An Evaluation of Three Techniques for Determining Young's Modulus of Mechanically Alloyed Materials"* Smith JS, Wyrrick MD and Poole JM, Dynamic Elastic Modulus Measurement in Materials, ASTM STP 1045, Wolfenden A Ed, ASTM Philadelphia, PA, 1990.

4.   *"Stahleisen-Sonderberichte Heft 10: Physikalische Eigenschaften von Stahlen und ihre Temperaturabhangogkeit Polynome und graphische Darstellungen"*; published by Verlag Stahleisen.

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APPENDIX A

MATHEMATICAL FORMULAE FOR CALCULATING UNCERTAINTIES IN DYNAMIC YOUNG'S MODULUS AT AMBIENT TEMPERATURE

In the first part, A1 to A5, of Appendix A we summarise the main relationships from the standard ASTM E1876-97 which are used in the calculation of dynamic modulus. A1 to A3 cover rectangular bars; A4 and A5 cylindrical bars. There then follows a discussion of how the uncertainties in individual measurements are likely to affect the uncertainties on reported measurands.

A1.   Dynamic Young's modulus of a rectangular bar

The dynamic Young's modulus is calculated from the following equation

$$E = 0.9465(mf_f^2 / b)(L^3 / t^3)T_1 \tag{A1}$$

where T_1 is a geometrical correction factor for the fundamental flexural mode to account for the bar's finite thickness and Poisson's ratio, and the other parameters are defined in Section 2 of this CoP. If $L/t > 20$ the geometrical effects are considered negligible and T_1 may be calculated directly from Equation (A2).

$$T_1 = [1.000 + 6.585(t / L)^2] \tag{A2}$$

If $L/t < 20$, then the value of T_1 should be calculated from Equation (A3).

$$T_1 = 1.000 + 6.585(1 + 0.0752\boldsymbol{m} + 0.8109\boldsymbol{m}^2)(t / L)^2 - 0.868(t / L)^4$$
$$- \frac{8.340(1 + 0.2023\boldsymbol{m} + 2.173\boldsymbol{m}^2)(t / L)^4}{1.000 + 6338(1 + 0.1408\boldsymbol{m} + 1.536\boldsymbol{m}^2)(t / L)^2} \tag{A3}$$

If Poisson's ratio, μ, is unknown an initial value must be assumed. It may be subsequently estimated by an iterative procedure from the calculation of shear modulus.

A2.   Dynamic shear modulus of a rectangular bar.

The dynamic Shear modulus is calculated from the following equation

$$G = 4(Lmf_t^2 / bt)[B / (1 + A)] \tag{A4}$$

Where A, and B are empirical width to thickness correction factors

$$B = \frac{b/t + t/b}{t(t/b) - 2.52(t/b)^2 + 0.21(t/b)^6} \tag{A5}$$

$$A = \frac{\left[0.5062 - 0.8776(b/t) + 0.3504(b/t)^2 - 0.0078(b/t)^3\right]}{\left[12.03(b/t) + 9.892(b/t)^2\right]} \tag{A6}$$

ASTM E1876-97 allows A to be omitted, noting that errors of less than 2% will result.

A3.   Poisson's ratio.

Assuming isotropic behaviour:

$$\boldsymbol{m} = (E/2G) - 1 \tag{A7}$$

A4.   Dynamic Young's modulus of a cylindrical bar.

For a cylindrical bar the dynamic Young's modulus is given by

$$E = 1.6067(mf_f^2)(L^3/D^4)T_1' \tag{A8}$$

where $E = 1.6067(mf_f^2)(L^3/D^4)T_1'$ is a correction factor for the fundamental flexural mode to account for the bar's finite thickness and Poisson's ratio. If $L/t > 20$, then $E = 1.6067(mf_f^2)(L^3/D^4)T_1'$ may be calculated directly from Equation (A9).

$$T_1' = [1.000 + 4.939(D/L)^2] \tag{A9}$$

If $L/D < 20$, then the value of T_1' should be calculated from Equation (A10).

$$T_1' = 1.000 + 4.939(1 + 0.0752\boldsymbol{m} + 0.8109\boldsymbol{m}^2)(D/L)^2 - 0.4883(D/L)^4$$
$$- \frac{4.691(1 + 0.2023\boldsymbol{m} + 2.173\boldsymbol{m}^2)(D/L)^4}{1.000 + 4.754(1 + 0.1408\boldsymbol{m} + 1.536\boldsymbol{m}^2)(D/L)^2} \tag{A10}$$

A5.   Dynamic shear modulus of a cylindrical bar.

For a cylindrical bar the dynamic Shear modulus is given by

$$G = 16mf_t^2(L/\boldsymbol{p}D^2) \tag{A11}$$

No empirical correction formulae are required for cylindrical bars.

A6. Sources of Uncertainty when Testing to ASTM E1876-97.

Turning now to the sources of uncertainty listed in the Main Procedure (Table 2) it is demonstrated how the typical uncertainties in the individual measurements, differences in apparatus etc. lead to each source's contribution to the error budget (sensitivity coefficient, c_i, multiplied by that measurement's uncertainty, u). It is assumed that the measuring equipment has been appropriately calibrated, and that a test procedure has been written following the standard which minimises typical measurement errors, and that the procedure is being applied by a trained operative. For simplicity, only the use of rectangular bars has been considered, and by way of illustration a mild steel bar of nominal dimensions 3mm x 15mm x 200mm is described.

A7. Calculation of Partial Derivatives.

The calculation of uncertainties in the reported values (E, G, m) on rectangular test pieces is made by calculating the "sensitivity coefficients" from the partial derivatives of equations (A1, A4 and A7) containing the measurands. When a derived quantity, Y, is a function of measurands x_1, x_2, x_3 …. , and each x_i is subject to uncertainty $u(x_i)$, then the resulting uncertainty in Y is given by:

$$U(Y)^2 = \sum \left(U(x_i) . \frac{\partial Y}{\partial x_i} \right)^2 \qquad (A12)$$

The partial derivatives for equation A1, which lead to the values of the sensitivity coefficients are given by:

$$\frac{\partial E}{\partial m} = \partial \frac{0.9465(mf_f^2 / b)(L^3 / t^3)T_1}{\partial m} = 0.9465(f_f^2 / b)(L^3 / t^3)T_1 = E . \frac{1}{m} \quad ;$$

hence $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad c_i = 1 \qquad (A13.1)$

similarly:

$$\frac{\partial E}{\partial f_f} = \partial \frac{0.9465(mf_f^2 / b)(L^3 / t^3)T_1}{\partial f_f} = \frac{2E}{f_f} \quad ; c_i = 2 \qquad (A13.2)$$

$$\frac{\partial E}{\partial b} = \partial \frac{0.9465(mf_f^2 / b)(L^3 / t^3)T_1}{\partial b} = -\frac{E}{b} \quad ; \ c_i = 1 \qquad (A13.3)$$

$$\frac{\partial E}{\partial L} = \partial \frac{0.9465(mf_f^2 / b)(L^3 / t^3)T_1}{\partial L} = \frac{3E}{L} \quad ; \ c_i = 3 \qquad (A13.4)$$

$$\frac{\partial E}{\partial t} = \partial \frac{0.9465(mf_f^2 / b)(L^3 / t^3)T_1}{\partial t} = -\frac{3E}{t} \quad ; \ c_i = 3 \qquad (A13.5)$$

Thus,

$$U(E)^2 = E^2\left[\left(\frac{U(m)}{m}\right)^2 + \left(2.\frac{U(f_f)}{f_f}\right)^2 + \left(-\frac{U(b)}{b}\right)^2 + \left(3.\frac{U(L)}{L}\right)^2 + \left(-3.\frac{U(t)}{t}\right)^2\right]$$

(A14)

Strictly speaking, the correction factor, T_1, also contains the measurements t and L, and should have been differentiated in Equations A13.4 and A13.5. However, with a typical value of (L/t) of 20, the contribution of the second term in Eq. A2 is 0.0165. Thus, the uncertainty in both t, and L is negligible in calculating the uncertainty in T_1, and has therefore been neglected.

Simplifying Equation A14, by expressing the uncertainties as a percentage of the measurand (eg. u(t)% = 100×u(t)/t etc.) and acknowledging that the sensitivity coefficients will have absolute values, then we have:

$$U(E)\% = \left[(1.U(m)\%)^2 + (2.U(f_f)\%)^2 + (1.U(b)\%)^2 + (3.U(L)\%)^2 + (3.U(t).\%)^2\right]^{0.5}$$

(A15)

Equations A4 and A7 are similarly used to estimate the percentage uncertainties in G and m, as shown in equations A16 and A17, respectively.

$$U(G)\% = \left[(1.U(m)\%)^2 + \left(2.U(f_t)\%\right)^2 + (1.U(b)\%)^2 + (1.U(L)\%)^2 + (1U(t).\%)^2\right]^{0.5}$$

(neglecting the negligible contributions via A,B) (A16)

$$U(\mathbf{m})\% = \left[(1.U(E)\%)^2 + \left(1.U(G)\%\right)^2\right]^{0.5}$$

(A17)

Equations A15, A16 and A17, therefore, form the basis of calculating the combined uncertainty, U_c, in Table 3 of the Main Procedure.

A8.   Uncertainty due to Measuring the Mass and Dimensions of the Test Bar.

In general, it is to be expected that the width and thickness of the test piece will be measured using a micrometer. Typical uncertainties due to the accuracy of the micrometer, uniformity of the test piece and skill of the operator can amount to uncertainties of ±0.00001m, which with a minimum practical thickness of about 0.003m results in an uncertainty of 0.3% contributing through t^{-3} in Equation A1. Some of this uncertainty can be reduced if larger thicknesses are used, since the errors in the micrometer readings and in the shape tolerance of machined test pieces are generally fixed. Typically, the width of a rectangular test piece is 5 times the thickness, and hence the uncertainties in its measurement are much less. Generally, calipers will be used to measure the length, and uncertainties of ±0.0005m are typical. This corresponds to 0.25% on a length of 0.200m, contributing through L^3 in Equation A1.

Generally, on well-machined test bars the main sources of uncertainty in dimensional measurements are due to the tolerances of the micrometer and calipers, which are Type B errors, and cannot be reduced by repeated measurements. On the other hand, the laboratory should consider the need to perform repeated measurements on test bars whose dimensions vary by more than tolerance of the measuring instruments.

The shape tolerance will not only contribute to uncertainties in the measurement of dimensions, but also in the measurement of frequency, as different frequencies can be induced in non-uniform bars. A round-robin study (Reference 3) reported in ASTM E1876-97 on a machined silicon nitride bar and an alumina bar which was not machined, and had a thickness variation of 0.0045 to 0.0048m along its length, indicated that the non-uniformity of the alumina bar may have contributed about 0.4% to the uncertainty in frequency. Similarly, rounded edges and chamfers may cause changes in frequencies. However, both these effects can usually be considered negligible for machined bars.

Generally, the mass of a test piece can be measured with an uncertainty of $\pm$0.0002kg, which for a 0.070kg sample corresponds to 0.29%, contributing through m in Equation A1. This uncertainty is almost entirely due to the tolerance in the scales, and is therefore Type B.

A9.   Uncertainty due to measuring frequency.

With proper experimental technique, damping from the supports or the location of the transducer can be minimised, and can be considered negligible. (Damping causes a rise in frequency, which biases the reported result.) On a fundamental resonant frequency of 6000Hz the uncertainty (5 repeated readings) is reported in ASTM E1876-97 to be about 18Hz (or about 1.0% assuming a rectangular distribution on a typical frequency of 1,000Hz), which contributes through the f_f term in Equation A1.   This uncertainty is comprised principally of tolerances in the measuring equipment, and sometimes its resolution, and is relatively difficult to quantify except with calibrated signal generators and/or with round-robin investigations.

Generally, the repeatability of frequency measurements on a single test bar contributes a much smaller uncertainty than that quoted in the standard. This indicates that the combined uncertainty in measuring frequency has as its major contribution tolerances in the test set-up, and it is therefore included as a Type B uncertainty in Table 3.

A10.   Uncertainty due to environment.

Some materials such as certain composites and some rocks are affected by humidity which can lead to a change the fundamental resonant frequency, or a change in the mass. However, these influences are considered negligible for most metallic materials. Similarly, ASTM E1876-97 requires that the test temperature at measurement be recorded.   In most laboratories the uncertainty in this measurement is likely to be better than $\pm$3°C. Its influence is through variations in the test piece dimensions (during measurement of those dimensions, and during measurement of

frequencies). Modulus values are also dependent on temperature, but small variations ($\pm 3°C$) in temperature can usually be ignored.

A11.   Uncertainty due to the operator, procedure and assumptions.

As described above, the evaluated dynamic Young's modulus and shear modulus can be used to estimate Poisson's ratio. ASTM E1876-97 describes an iterative procedure, stating that the Poisson's ratio is that when the difference between the current and previous estimate is "less than 2%". This can give rise to a 2% uncertainty with a rectangular distribution. No guidance is given for the starting value of Poisson's ratio, and hence the final value can be approached either from below or above.

Note also that if the correction factor A is omitted from Equation A4, then a systematic error u_A (most often small and positive) must be added to the combined uncertainty in the following fashion:

$$U(G)\% = U_A\% + \left[(1.U(m)\%)^2 + \left(2.U(f_t)\%\right)^2 + (1.U(b)\%)^2 + (1.U(L)\%)^2 + (1.U(t).\%)^2 \right]^{0.5}$$

(A18)

The assumption of isotropic properties in calculating Poisson's ratio is also considered to be true. However, the test laboratory should always be alert to significant anisotropy in metals whose structure is heavily textured, in geological samples containing porosity etc.

A12.   Uncertainty of the mean value of E (repeatability).

Repeatability is a Type A uncertainty contribution. It is the standard deviation of the estimated mean value of a series of test results on different bars of the same material under the same conditions considered in the uncertainty analysis.

$$u_{rep} = \frac{1}{\sqrt{n}} \left[\sqrt{ \frac{1}{n-1} \sum_{1}^{n} \left(E_i - \overline{E} \right)^2 } \right]$$

(A19)

where $\overline{E}$ is the mean dynamic Young's modulus. Obviously, if only one test is conducted then the repeatability should not be included in the calculations, and the reported uncertainty is *that of a single test*.

The determination of dynamic Young's modulus is non-destructive, and a series of tests on the same test bar, say over a period of weeks with different operators, would give useful information about the repeatability of the test method without any influence of material variability. However, the customer is generally more interested in determining the uncertainty in dynamic Young's modulus for several test bars, either from the same batch or for different batches of material to the same grade. (The test laboratory should take due care to ensure that the test bars have been

sampled to be representative of the batch or grade.) In the first case, the reported uncertainty is that of a mean value of *a batch of material*. In the second case, the reported uncertainty is that of a mean value of *a grade of material*.

A study of dynamic Young's Modulus of 60 unalloyed and alloyed steel grades found that repeatability was about $\pm 0.3\%$ (thought to be with approximately a 95% coverage factor) over the temperature 20 to 600°C range, Ref. [4].

APPENDIX B

A WORKED EXAMPLE FOR CALCULATING UNCERTAINTIES IN DYNAMIC YOUNG'S MODULUS OF A RECTANGULAR BAR

B1. Introduction.

A customer asked the testing laboratory to carry out dynamic modulus, shear modulus and Poisson's ratio measurements on a single, rectangular steel bar (marked: XYZ123), of nominal dimensions 0.003m x 0.015mx 0.200m, according to ASTM E1876-97. The laboratory has considered the tolerances in its test facility, and found that the sources of uncertainty were identical to those described in Table 2 of this CoP

B2. Estimation of Input Quantities to the Uncertainty Analysis.

1 All measurements were performed according to the laboratory's own written test procedure, and were made within the controlled temperature ($21\pm2°$C) of the laboratory, using appropriately calibrated instruments.

2 The laboratory has at its disposal a micrometer with a tolerance of $\pm$0.01mm, which was used to measure the bar's thickness and width, and a caliper with a tolerance of $\pm$0.5mm was used to measure the length. The laboratory made repeat measurements of the bar's thickness width and length, demonstrating that the shape tolerances were smaller than can be detected with the micrometer or caliper.   A measuring scale with a tolerance of 0.2g was used to measure the mass of the bar.

3 A piezoelectric transducer / oscilloscope system was used to measure the resonant frequencies. The measuring system had been checked in a round-robin study with other laboratories, and the laboratories' declared uncertainty of $\pm$18Hz on the measured frequency was believed to apply. Repeat measurements each of the fundamental frequencies in tension and in torsion were made, and the mean taken.

B3. Preliminary Calculations

The following calculations use mean values, except where stated. In the first instance, a number of calculations were performed to check the density, and provide measurements of length to thickness and width to thickness ratios, as follows

$$r = m / (L.b.t) \qquad = 7836.6 \text{ kg m}^{-3} \qquad \text{(B1)}$$
$$L/t \qquad = 39.67 \qquad \text{(B2)}$$
$$b/t \qquad = 5.006 \qquad \text{(B3)}$$

Calculation of the area at various positions along the bar confirmed that the standard deviation divided by the mean is 0.19% and therefore the test bar was within the permitted shape tolerance.

B4.   Calculations of Young's Modulus, Shear Modulus and Poisson's Ratio.

The Young's modulus was calculated from Equation A1. The length to thickness ratio was greater than 20, and the correction factor, T_1, estimated by Equation A3 (initial Poisson's Ratio of 0.3) and Equation. A2, is 1.004577953 and 1.004182875, respectively. Although the difference was small, the value from Equation A3 was specified in the laboratory's procedure, and therefore the initial estimate of the Young's modulus was found to be:

$$E_{\text{(initial)}} = 206.1 \text{ GPa}$$

Shear modulus was calculated from Equation A4, using values for the correction factors B and A calculated from Equations A5 and A6,   (whilst ASTM E1876-97 allows A to be omitted, a positive systematic error in G of 1.3% would have resulted.), giving

$$G = 78.53 \text{ GPa}$$

Poisson's ratio was then estimated using Equation A7, giving $\mu = 0.3123$, and compared with the initial value of 0.3. An iterative method was then used to adjust the initial value until it coincided with that estimated by Equation A7. In fact, as the value of Young's Modulus is insensitive to Poisson's ratio when the length to thickness ratio is greater than 20
modulus and Poisson's ratio were found to be:

$E_{\text{(final)}}$ $= 206.1 \text{ GPa}$
$\boldsymbol{m}_{\text{(final)}}$ $= 0.312$

Table B1. Measurands, Measurements, their Units and Symbols within ASTM E1876 -97

Measurands	Symbol	Values	Mean
Dynamic Young's modulus, Pa	E	-	-
Dynamic shear modulus, Pa	G	-	-
Poisson's ratio, (dimensionless)	µ	–	–
Mass of the bar, kg	m	0.04214 0.04225 0.04222 0.04213 0.0421	0.042168
Width of the bar, m (Note 1)	b	0.01504 0.01501 0.01507 0.01501 0.01505	0.015036
Length of the bar, m (Note 1)	L	0.1192 0.1193 0.1191 0.1191 0.1191	0.11916
Thickness of the bar, m (Note 1)	t	0.0030024 0.0030054 0.0030048 0.0030012 0.0030024	0.00300324
Fundamental resonant frequency of a rectangular bar, or circular rod, in flexure, Hz	f_f	1114 1113 1112 1112 1111	1112.4
Fundamental resonant frequency of rectangular bar in torsion, Hz	f_t	1007 1006 1005 1005 1004	1005.4
Temperature at time of measurements, °C			21

Notes:

1. Measurements were made at five equally spaced positions along the length of the bar. The measurement positions for width and thickness correspond, so that an area calculation can be performed to test for shape tolerance.

2. Young's modulus, shear modulus and Poisson's ratio are obtained by calculation

Table 3a. Worksheet for Uncertainty Budget Calculations For Estimating the Uncertainty in Dynamic Young's Modulus of a Rectangular Bar

Source of uncertainty	Symbol	Value	Probability distribution	Divisor d_v	c_i	$u_I(E)$ $\pm$GPa	v_i or v_{eff}
Mass of the bar	m	$\pm$0.2g	rectangular	$\sqrt{3}$	1	0.56	∞
Width of the bar	b	$\pm$0.01mm	rectangular	$\sqrt{3}$	1	0.08	∞
Length of the bar	L	$\pm$0.5mm	rectangular	$\sqrt{3}$	3	1.50	∞
Thickness of the bar	t	$\pm$0.01mm	rectangular	$\sqrt{3}$	3	1.19	∞
Fundamental frequency in flexure	f_f	18Hz	rectangular	$\sqrt{3}$	2	3.85	∞
Mean value of E	$u(E)_{rep}$	206.1	normal	1	1	-	n-1
Combined standard uncertainty	u_c		normal			4.34	$\boldsymbol{n_{eff}}$
Expanded uncertainty (k = 2)	U		normal			8.68	$\boldsymbol{n_{eff}}$

1) Calculations based on Equation A15
2) The incomplete iterative solution of **_m_** will have a computable but negligible effect on *E* through Equation A3.

Table 3b. Worksheet for Uncertainty Budget Calculations For Estimating the Uncertainty in Shear Modulus of a Rectangular Bar

Source of uncertainty	Symbol	Value	Probability distribution	Divisor d_v	c_i	$u_i(G)$ Pa	v_i or v_{eff}
Mass of the bar	m	$\pm$0.2g	rectangular	$\sqrt{3}$	1	0.21	∞
Width of the bar	b	$\pm$0.01mm	rectangular	$\sqrt{3}$	1	0.03	∞
Length of the bar	L	$\pm$0.5mm	rectangular	$\sqrt{3}$	1	0.19	∞
Thickness of the bar	t	$\pm$0.01mm	rectangular	$\sqrt{3}$	1	0.15	∞
Fundamental frequency in torsion	f_t	18Hz	rectangular	$\sqrt{3}$	2	1.62	∞
Mean value of G	$u(G)_{rep}$	78.5	normal	1	1	-	n-1
Combined standard uncertainty	U_c		normal			1.66	$\boldsymbol{n_{eff}}$
Expanded uncertainty	U		normal			3.32	$\boldsymbol{n_{eff}}$

1) Calculations based on Equation A16
2) The omission of the correction factor A is for most practical purposes a bias, resulting in an overestimate of *G*.

Table 3c. Worksheet for Uncertainty Budget Calculations For Estimating the Uncertainty in Poisson's Ratio of a Rectangular Bar

Source of uncertainty	Symbol	Value	Probability distribution	Divisor d_v	c_I	$u_i(m)$	v_i or v_{eff}
Dynamic Young's modulus	E	4.34GPa	normal	1	1	0.0065	∞
Dynamic Shear modulus	G	1.66GPa	normal	1	1	0.0066	∞
Mean value of m	$u(m)_{rep}$	0.312	normal	1	1	-	n-1
Combined standard uncertainty	U_c		normal			0.0093	n_{eff}
Expanded uncertainty	U		normal			0.0186	n_{eff}

1) Calculations based on Equation A17

B5.   Uncertainty Calculations and Reporting of Results.

The mean values from Table B1 and the tolerances described in the text were entered into the Uncertainty Calculation Worksheet, Tables B2a, B2b and B2c. Note that in performing this test, the laboratory included an estimate of the correction factor, A, in its calculation of G, and also completed the iterative solution of μ, thereby removing two systematic sources of uncertainty. The repeated measures could, in theory, be used to estimate Type A errors for the measurands. However, the laboratory chose to use the Type B errors recorded in Table B2, as the tolerances on the test devices exceeded those errors contributed by repeated measures.

The test laboratory was supplied with a single test bar, and therefore could perform the uncertainty calculation solely for that test bar, and not for a batch or grade of material.

The laboratory presents the results in the form shown below.

> *The results of a test conducted according to ASTM E1876-97 on a steel test piece reference XYZ123 are:*
>
> *Dynamic Young's Modulus   206.1 ± 8.7 GPa*
> *Dynamic Shear Modulus   78.5 ± 3.3 GPa*
> *Poisson's Ratio   0.312 ± 0.019*
>
> *The above reported expanded uncertainty is based on a standard uncertainty multiplied by a coverage factor k=2, providing a level of confidence of approximately 95%. The uncertainty evaluation was carried out in accordance with UNCERT COP 13: 2000.*