A Brief Workshop on Measurement Uncertainty

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Uncertainties: The Easy Solution

“If your experiment needs statistics, you should have done a better experiment”

Ernest Rutherford (1871 - 1937)
Uncertainties

• Principle
  – There is some uncertainty about every measurement result … a “margin of doubt”.

• What confidence do WE have in our values?
• What confidence do OTHERS have in our values?
• Can our values withstand objective scrutiny, or be defended in a COURT OF LAW?
Measurement Uncertainty : Basic Concepts

• A measurement result is incomplete without a statement of the associated uncertainty

• When you know the uncertainty in a measurement, then you can judge its fitness for purpose

• Good measurement practice can reduce uncertainty

• Understanding measurement uncertainty is the first step to reducing it.
Where do errors and uncertainties come from?

- The measuring instrument: bias, drift, noise …
- The condition being measured, which may not be stable
- The measurement process itself
- “Imported” uncertainties
  - Calibration uncertainties (Better than not calibrating at all)!
- Operator skill
- Environmental conditions (temperature, pressure, humidity etc…)
- Others …
Literature

**GPG 11**

*A Beginner’s Guide to Uncertainty of Measurement*

**NPL 2001**
GPG 49
The Assessment of Uncertainty in Radiological Calibration and Testing
NPL 2003
M3003  EDITION 2 | JANUARY 2007

The Expression of Uncertainty and Confidence in Measurement

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Provides internationally agreed concepts, recommendations, procedures for uncertainty evaluation.

- Underlying philosophy
- Means, standard uncertainties
- Distributions
- Sensitivity coefficients
- Combination of uncertainties
- Uncertainty budgets
Measurement Uncertainty

• From the “GUM”
  – “parameter associated with the result of a measurement, that characterises the dispersion of the values that could reasonably be attributed to the measurand”.
    • i.e.: a standard deviation or the half width of an interval having a stated level of confidence
    • the measurement result is the “best estimate” of the value of the measurand, and all components of uncertainty contribute to the dispersion.

• Simplified definition
  – quantified doubt about the measurement result.
GUM : Basic Concepts

- Identify quantities that influence the measurand
- Develop a model to account for the inter-relations of the input quantities that influence the measurand

\[ Y = f(X_1, X_2, \ldots, X_N) \]
Knowledge of each input quantity is incomplete, and must be expressed as a probability density function (PDF).

The expectation of the PDF is taken as the best estimate of the value for that quantity.

The standard deviation of the PDF is taken as the standard uncertainty associated with that estimate.

The PDF is based on knowledge of the input quantity that may be inferred from repeated measurements (TYPE A) or scientific judgement based on all available information on the variability of that quantity (TYPE B).
GUM: Basic Concepts

\[ Y = f(X_1, X_2, \ldots, X_N) \]

\[ x_1, u(x_1) \]

\[ x_2, u(x_2) \]

\[ \ldots \]

\[ x_N, u(x_N) \]

\[ F(X_1) \]

\[ F(X_2) \]

\[ F(X_N) \]
Steps to Evaluate Measurement Uncertainty

• Identify input quantities to measurement model
• Estimate uncertainties on inputs
  – (including probability distributions : form of the spread of results)
• Consider correlations between input parameters
• Establish sensitivity coefficients
  – How the output changes with respect to changes in the inputs
• Calculate standard uncertainty on output quantity for each input quantity
• Combine standard uncertainties
  – Propagation of Errors
  – Central Limit Theorem (combined uncertainty distribution)
• Calculate effective degrees of freedom
• Report expanded uncertainty at k =2
Evaluating Uncertainty

Two ways to evaluate individual components:

*Type A evaluations* - uncertainty estimates using statistics (usually from repeated readings)

*Type B evaluations* - uncertainty estimates from any other information, e.g. from

- past experience of the measurements
- from calibration certificates
- manufacturer’s specifications
- from calculations
- from published information
- and from common sense, experience, feel …

Not necessarily “random versus systematic”
Basic Statistics: repeat measurements

From repeated measurements you can:

- Calculate an average or mean
  - to get a better estimate of the “true” value
    \[ \bar{x} = \frac{\sum_{i=1}^{n} x_i}{n} \]

- Estimate the standard deviation
  - quantifies how different individual readings typically are from the mean of the whole set
    \[ s = \sqrt{\frac{\sum_{i=1}^{n} (x_i - \bar{x})^2}{n-1}} \]

- Estimate the standard deviation of the mean (SDOM)
  - estimate of the standard deviation of the distribution of means that would be obtained if the mean were measured many times.
    \[ s(\bar{x}) = \frac{s}{\sqrt{n}} = \sqrt{\frac{\sum_{i=1}^{n} (x_i - \bar{x})^2}{n(n-1)}} \]

- NB: “True” standard deviation can only be found from an infinite set of readings.
Basic Statistics : PDFs

For a continuous random variable $X$, with probability density function $f(x)$

**Expectation**: \[ E(X) = \mu_X = \int_{-\infty}^{+\infty} x \cdot f(x) \, dx \] 1st Ordinary Moment

**Variance**: \[ \text{Var}(X) = E\left(\left[ X - \mu_X \right]^2 \right) \] 2nd Central Moment

\[ = E(X^2) - 2 \mu_X^2 + \mu_X^2 \]
\[ = E(X^2) - E(X)^2 \]

For jointly distributed random variables $X$ and $Y$, with expectations $E(X) = \mu_X$ and $E(Y) = \mu_Y$

**Covariance**: \[ \text{Cov}(X,Y) = E\left[(X - \mu_X)(Y - \mu_Y)\right] \]
\[ = E(XY - X\mu_Y - Y\mu_X + \mu_X\mu_Y) \]
\[ = E(XY) - E(X)\mu_Y - E(Y)\mu_X + \mu_X\mu_Y \]
\[ = E(XY) - E(X)E(Y) \]
Probability Distributions

- Describe the form of the spread of results
  - Normal (Gaussian)
    - Most common
  - Rectangular (Uniform)
    - Results evenly spread
  - Triangular
    - Min, max and “most likely” known
  - U-shaped
    - Most likely at ends of distribution
Rectangular Distribution

\[ f(x) = \begin{cases} \frac{1}{b-a} & a \leq x \leq b \\ 0 & \text{otherwise} \end{cases} \]

\[ E[x] = \text{mean} = \int_a^b x \cdot \frac{1}{b-a} \, dx = \frac{1}{b-a} \left[ \frac{x^2}{2} \right]_a^b = \frac{b^2 - a^2}{2(b-a)} = \frac{(b+a)(b-a)}{2(b-a)} = \frac{a+b}{2} \]

\[ E[x^2] = \int_a^b x^2 \cdot \frac{1}{b-a} \, dx = \frac{1}{b-a} \left[ \frac{x^3}{3} \right]_a^b = \frac{a^2 + ab + b^2}{3} \]

\[ E[x^2] - E[x]^2 = \text{Variance} = \frac{a^2 + ab + b^2}{3} - \left( \frac{a^2 + 2ab + b^2}{4} \right) = \frac{(b-a)^2}{12} \]

\[ \sigma_x = \frac{(b-a)}{2\sqrt{3}} \]
Triangular Distribution

\[ f(x) = \begin{cases} \frac{4}{(b-a)^2}(x-a) & a \leq x \leq \frac{a+b}{2} \\ \frac{-4}{(b-a)^2}(x-b) & \frac{a+b}{2} \leq x \leq b \\ 0 & \text{otherwise} \end{cases} \]

\[ E[x] = \text{mean} = \frac{4}{(b-a)^2} \left[ \int_a^{\frac{a+b}{2}} x(x-a) \, dx + \int_{\frac{a+b}{2}}^b x(x-b) \, dx \right] = \frac{a+b}{2} \]

\[ E[x^2] = \frac{4}{(b-a)^2} \left[ \int_a^{\frac{a+b}{2}} x^2(x-a) \, dx + \int_{\frac{a+b}{2}}^b x^2(x-b) \, dx \right] = \frac{7a^2 + 10ab + 7b^2}{24} \]

\[ E[x^2] - E[x]^2 = \text{Variance} = \frac{7a^2 + 10ab + 7b^2}{24} - \frac{(6a^2 + 12ab + 6b^2)}{24} = \frac{(b-a)^2}{24} \]

\[ \sigma_x = \frac{(b-a)}{2\sqrt{6}} \]
U-Shaped Distribution

\[ f(x) = \begin{cases} \frac{-2}{(b-a)^2}(2x-(a+b)) & a \leq x \leq \frac{a+b}{2} \\ \frac{2}{(b-a)^2}(2x-(a+b)) & \frac{a+b}{2} \leq x \leq b \\ 0 & \text{otherwise} \end{cases} \]

\[
E[x] = \text{mean} = \frac{-2}{(b-a)^2} \left[ \int_a^{a+b/2} x(2x-(a+b)) \, dx - \int_{a+b/2}^b x(2x-(a+b)) \, dx \right] = \frac{a+b}{2}
\]

\[
E[x^2] = \frac{-2}{(b-a)^2} \left[ \int_a^{a+b/2} x^2(2x-(a+b)) \, dx - \int_{a+b/2}^b x^2(2x-(a+b)) \, dx \right] = \frac{3a^2 + 2ab + 3b^2}{8}
\]

\[
E[x^2] - E[x]^2 = \text{Variance} = \frac{3a^2 + 2ab + 3b^2}{8} - \frac{(2a^2 + 4ab + 2b^2)}{8} = \frac{(b-a)^2}{8}
\]

\[
\sigma_x = \frac{(b-a)}{2\sqrt{2}}
\]
Equivalent probability

Uncertainties are expressed in terms of equivalent probability

The divisor “normalises components to the same probability”
Central Limit Theorem

The combined uncertainty distribution is approximately **Normal**

Except when one uncertainty contribution **DOMINATES** all others:

the combined distribution is essentially that of this dominant contribution

this single dominant uncertainty could be non-normal
How many values within one standard deviation? (Normal)

For normal probability distribution, the probability that a measurement will fall within one standard deviation is 67%
Normal distribution

Sample standard deviation $s$ is an estimate of population standard deviation $\sigma$

$k$ is "coverage factor"

$1\sigma$, 68% confidence, $k=1$

$2\sigma$, 95% confidence, $k=2$
How many values within $k$ standard deviations? (Normal)

<table>
<thead>
<tr>
<th>$k$</th>
<th>0.25</th>
<th>0.5</th>
<th>0.75</th>
<th>1.0</th>
<th>1.25</th>
<th>1.5</th>
<th>1.75</th>
<th>2.0</th>
<th>2.5</th>
<th>3.0</th>
<th>3.5</th>
<th>4.0</th>
</tr>
</thead>
<tbody>
<tr>
<td>Prob (%)</td>
<td>20</td>
<td>38</td>
<td>55</td>
<td>67</td>
<td>79</td>
<td>87</td>
<td>92</td>
<td>95.2</td>
<td>98.8</td>
<td>99.7</td>
<td>99.95</td>
<td>99.99</td>
</tr>
</tbody>
</table>
Significance of probability distributions

• Once we have determined the shape of the output probability distribution (typically normal), we can calculate which part of the values from the set of measurement results will fall between any two chosen values.

• These values determine the **interval**, or the margin of uncertainty. The number showing the fraction of all results falling into this interval is the **confidence level**.
Coverage factor

- **Confidence level** is a number expressing degree or level of confidence in the result. Usually expressed as percentage or as a number of standard deviations.
- To express overall uncertainty at another level of confidence, re-scaling by a **coverage factor $k$** is needed.
Coverage factor

<table>
<thead>
<tr>
<th>Distribution</th>
<th>divisor</th>
<th>Confidence level</th>
</tr>
</thead>
<tbody>
<tr>
<td>Normal</td>
<td>1.0</td>
<td>67.7%</td>
</tr>
<tr>
<td>Normal</td>
<td>2.0</td>
<td>95.5%</td>
</tr>
<tr>
<td>Normal</td>
<td>3.0</td>
<td>99.7%</td>
</tr>
<tr>
<td>Rectangular</td>
<td>$\sqrt{3}$</td>
<td>100%</td>
</tr>
<tr>
<td>Triangular</td>
<td>$\sqrt{6}$</td>
<td>100%</td>
</tr>
</tbody>
</table>
Uncertainty of a computed quantity

Suppose quantity $A$ is to be computed from two measured quantities, $N$ and $t$ by means of a theoretical formula.

We know uncertainty in $N$ and uncertainty in $t$. What is the uncertainty in $A$?
Uncertainty Propagation

- How to calculate the uncertainty of a computed quantity
- differential calculus is needed
- combined uncertainty: are the contributing uncertainties independent or non-independent?
- General way of combining uncertainties involves sensitivity coefficients
Law of Propagation of Uncertainty
(no correlations : independent)

**absolute** \[ u_c(y) = \sqrt{\sum u_i^2(y)} \]

**relative** \[ \frac{u_c(y)}{y} = \sqrt{\sum \left[ \frac{u_i(y)}{y} \right]^2} \]

Each \( u_i(y) \) component may be expressed as

\[ u_i(y) = c_i u(x_i) \]

or \[ \frac{u_i(y)}{y} = c_i^{rel} \frac{u(x_i)}{x_i} \]

**absolute** \[ u_c(y) = \sqrt{\sum c_i^2 u^2(y)} \]

**relative** \[ \frac{u_c(y)}{y} = \sqrt{\sum \left[ c_i^{rel} \frac{u(x_i)}{x_i} \right]^2} \]
Law of Propagation of Uncertainty: \( y = f(\ldots x_i \ldots) \)

*Deviations from mean values:*

\[
\delta y = \sum \delta y_i = \sum \left( \frac{\partial f}{\partial x_i} \right) \delta x_i = \sum c_i \delta x_i
\]

\[
\delta y^2 = \sum c_i^2 \delta x_i^2 + \sum \sum c_i c_j \delta x_i \delta x_j
\]

\[
E(\delta y^2) = u_c^2(y)
\]

\[
u_c(y) = \sqrt{\sum c_i^2 u^2(x_i) + \sum \sum c_i c_j \text{cov}(x_i, x_j)}
\]
Combining standard uncertainties

\[ u_c = \sqrt{u_1^2 + u_2^2 + u_3^2 + \ldots \text{etc.}} \]

(Summation in quadrature)

This rule applies where the result is found using addition or subtraction

Other versions of this rule apply where the model has multiplication or division or more complicated functions

All uncertainties must be in same units and same level of confidence
Combined Uncertainty

Once quantities $x_1, \ldots, x_N$ are measured and corresponding $u_1, u_2, \ldots, u_N$ determined, the standard uncertainty of the output quantity $y$ is:

$$ u(y) = \sqrt{c_1^2 u^2(x_1) + c_2^2 u^2(x_2) + \ldots + c_N^2 u^2(x_N)}. $$

Where

$$ c_1 = \frac{\partial y}{\partial x_1}, \quad c_2 = \frac{\partial y}{\partial x_2}, \quad c_N = \frac{\partial y}{\partial x_N}, $$

are sensitivity coefficients.
Combining Uncertainties: Absolute or Relative?

\[ y = 2a + b \]

\[ c_a = \frac{\partial y}{\partial a} = 2 \quad ; \quad c_b = \frac{\partial y}{\partial b} = 1 \]

**absolute**

\[ u^2(y) = 2^2 u^2(a) + u^2(b) \]

**relative**

\[ \left( \frac{u(y)}{y} \right)^2 = \frac{2^2 a^2}{(2a+b)^2} \left( \frac{u(a)}{a} \right)^2 + \frac{b^2}{(2a+b)^2} \left( \frac{u(b)}{b} \right)^2 \]
Combining Uncertainties: Absolute or Relative?

\[ y = 2ab \]

\[ c_a = \frac{\partial y}{\partial a} = 2b \quad ; \quad c_b = \frac{\partial y}{\partial b} = 2a \]

**absolute**

\[ u^2(y) = 4b^2u^2(a) + 4a^2u^2(b) \]

**relative**

\[
\left( \frac{u(y)}{y} \right)^2 = \frac{4b^2a^2}{(2ab)^2} \left( \frac{u(a)}{a} \right)^2 + \frac{4a^2b^2}{(2ab)^2} \left( \frac{u(b)}{b} \right)^2 \\
= \left( \frac{u(a)}{a} \right)^2 + \left( \frac{u(b)}{b} \right)^2
\]
General rule for combining independent uncertainties

V is the quantity to be calculated from measurements of x and y. Uncertainties in x and y are $u_x$ and $u_y$ respectively.

The combined uncertainty $u_V$ is:

$$u_V = \sqrt{\left( \frac{\partial V}{\partial x} \right)^2 (u_x)^2 + \left( \frac{\partial V}{\partial y} \right)^2 (u_y)^2}$$
Combining independent Uncertainties.

Example

Calculate kinetic energy of an object, whose mass is measured as $m=0.230\pm0.001\text{kg}$ and velocity is $v=0.89\pm0.01\text{m/s}$.

\[
E = \frac{1}{2}mv^2
\]

\[
\frac{1}{2}mv^2 = \frac{0.230 \cdot (0.89)^2}{2} = 0.091
\]

\[
u_E = \sqrt{\left(\frac{\partial E}{\partial m}\right)^2 (u_m)^2 + \left(\frac{\partial E}{\partial v}\right)^2 (u_v)^2}
\]
Combining independent Uncertainties. Example

\[
\frac{\partial E}{\partial m} = \frac{\partial}{\partial m} \left( \frac{1}{2} m v^2 \right) = \frac{1}{2} v^2 \ \frac{\partial}{\partial m} m = \frac{1}{2} v^2
\]

\[
\left( \frac{\partial E}{\partial m} \right)^2 = \frac{v^4}{4} = 0.63
\]

\[
\frac{\partial E}{\partial v} = \frac{\partial}{\partial v} \left( \frac{1}{2} m v^2 \right) = \frac{1}{2} m \ \frac{\partial}{\partial v} v^2 = \frac{1}{2} m 2v = mv
\]

\[
\left( \frac{\partial E}{\partial v} \right)^2 = 0.04
\]

\[
u_E = \sqrt{0.63 \cdot (0.001)^2 + 0.04 \cdot (0.01)^2} = 0.002
\]

\[
E = 0.091 \pm 0.002 \quad \left[ \frac{kg \cdot m^2}{s^2} = J \right]
\]
General rule for combining correlated uncertainties

Suppose $x$ and $y$ are completely correlated

(only possible when experimental uncertainty is negligible and the quantities are in error due to some common cause eg. Incorrectly calibrated instrument)

$$u_V = \frac{\partial V}{\partial x} u_x + \frac{\partial V}{\partial y} u_y$$
Graphical representation of combining uncertainties

Independent uncertainties

Correlated uncertainties
A more realistic example

\[
A := C \left( \frac{V_s}{t_s} - \frac{V_{bg}}{t_{bg}} \right) \cdot \text{VC} \cdot \frac{DF}{CF}
\]

\[
\left( \frac{d}{dC} A \right) \rightarrow \left( \frac{V_s}{t_s} - \frac{V_{bg}}{t_{bg}} \right) \cdot \text{VC} \cdot \frac{DF}{CF}
\]

\[
\left( \frac{d}{dV_s} A \right) \rightarrow \frac{C}{t_s} \cdot \text{VC} \cdot \frac{DF}{CF}
\]

\[
\left( \frac{d}{dV_{bg}} A \right) \rightarrow \frac{-C}{t_{bg}} \cdot \text{VC} \cdot \frac{DF}{CF}
\]

\[
\left( \frac{d}{dt_s} A \right) \rightarrow \frac{C}{\text{VC}} \cdot \frac{DF}{CF}
\]

\[
\left( \frac{d}{dV_{bg}} A \right) \rightarrow \frac{-C}{t_{bg}} \cdot \frac{DF}{CF}
\]

\[
\left( \frac{d}{dV_C} A \right) \rightarrow \frac{DF}{CF^2}
\]

\[
\left( \frac{d}{dDF} A \right) \rightarrow \frac{C}{\text{VC}} \cdot \frac{DF}{CF}
\]

\[
\left( \frac{d}{dCF} A \right) \rightarrow \frac{-C}{\text{VC}} \cdot \frac{DF}{CF^2}
\]
Practical tips for calculating sensitivity coefficients

Calculation of $c_1$, $c_2$, $\ldots$, $c_N$ by partial differentiation is often lengthy process. Alternatively:

- Change one of the input variables by a known amount
- Keep all other inputs constant
- Note the changes in the output quantity
Tips for calculating sensitivity coefficients

The amount by which you changed the input variable

\[ c = \frac{\partial y}{\partial x} \approx \frac{y(x + h) - y(x)}{h} \]
## Spreadsheet model ( "formulation" )

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Source of uncertainty</th>
<th>[uncert] Value ±</th>
<th>Probability distribution</th>
<th>Divisor $c_i$</th>
<th>Standard uncertainty $u ±$ units</th>
<th>$\nu_i$ or $\nu_{eff}$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td><em>One row for each contributing uncertainty</em></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$u_c$</td>
<td>Combined standard uncertainty</td>
<td>Normal</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$U$</td>
<td>Expanded uncertainty</td>
<td>Normal (k=2)</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Spreadsheet model

The columns are:

**Symbol or reference**

**Source of uncertainty** – brief text description of each uncertainty

**Value (±)** – estimate of the uncertainty – from what info you have, e.g. “worst case limits” or “standard deviation”. Show units, e.g. °C.

**Probability distribution** - rectangular, normal, (or rarely others)

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Source of uncertainty</th>
<th>[uncert] Value ±</th>
<th>Probability distribution</th>
<th>Divisor $c_i$</th>
<th>Standard uncertainty $u ±$ units</th>
<th>$v_i$ or $v_{eff}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$u_c$</td>
<td>Combined standard uncertainty</td>
<td></td>
<td>Normal</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$U$</td>
<td>Expanded uncertainty</td>
<td>$U$</td>
<td>Normal $(k=2)$</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Spreadsheet model

**Divisor** - factor to normalise a *value* to a *standard uncertainty* (depends on probability distribution).

**c_i** - *sensitivity coefficient* to convert to consistent units

**u** – *standard uncertainty* a “standardised measure of uncertainty” calculated from previous columns: \( u = \text{value} \div \text{divisor} \times c_i \).

**\( \nu_i \)** – *effective number of degrees of freedom* – to do with reliability of the uncertainty estimate (the uncertainty in the uncertainty!) - sometimes ignored! Sometimes infinite! (\( \infty \))
Spreadsheet model

The rows are:

**Title row**

**One row for each uncertainty**

**One row for combined standard uncertainty,** $u_c$, by summing “in quadrature” and taking square root, i.e.,

$$u_c = \sqrt{a^2 + b^2 + c^2 + \ldots}$$

**Final row showing expanded uncertainty,** $U = k \times u_c$. Normally coverage factor $k = 2$ (level of confidence of 95% percent, if many degrees of freedom).

Expanded uncertainty is what finally should be reported.
## Sources of uncertainty

### Type A uncertainties (evaluated by the statistical analysis of a series of observations)

<table>
<thead>
<tr>
<th>Term</th>
<th>Source of uncertainty</th>
<th>± ( u(x_i) )</th>
<th>± %</th>
<th>Distribution</th>
<th>Divisor</th>
<th>( c_i )</th>
<th>( u_i )</th>
<th>( v_i )</th>
</tr>
</thead>
<tbody>
<tr>
<td>SD</td>
<td>Repeatability</td>
<td>0.00554 pA</td>
<td>0.0727</td>
<td>Normal</td>
<td>1</td>
<td>1</td>
<td>0.0727</td>
<td>N-1</td>
</tr>
</tbody>
</table>

### Type B uncertainties (evaluated by means other than the statistical analysis of a series of observations)

<table>
<thead>
<tr>
<th>Term</th>
<th>Source of uncertainty</th>
<th>± ( u(x_i) )</th>
<th>± %</th>
<th>Distribution</th>
<th>Divisor</th>
<th>( c_i )</th>
<th>( u_i )</th>
<th>( v_i )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( C_{F_0} )</td>
<td>Calibration factor of the principal radionuclide</td>
<td>0.1042</td>
<td>1.00</td>
<td>Normal</td>
<td>1.00</td>
<td>1</td>
<td>1.00</td>
<td>∞</td>
</tr>
<tr>
<td>( C_{F_i} )</td>
<td>Calibration factor(s) of the contaminant radionuclide</td>
<td>N/A</td>
<td>N/A</td>
<td>Normal</td>
<td>1</td>
<td>N/A</td>
<td>N/A</td>
<td>∞</td>
</tr>
<tr>
<td>VC</td>
<td>Volume correction factor</td>
<td>0.003</td>
<td>0.003</td>
<td>Normal</td>
<td>1</td>
<td>1</td>
<td>0.003</td>
<td>∞</td>
</tr>
<tr>
<td>( X_i )</td>
<td>Fractional content of contaminant radionuclide</td>
<td>N/A</td>
<td>N/A</td>
<td>Normal</td>
<td>1</td>
<td>N/A</td>
<td>N/A</td>
<td>∞</td>
</tr>
<tr>
<td>( C_{f} )</td>
<td>Capacitance of the feedback capacitor</td>
<td>0.0125</td>
<td>0.0125</td>
<td>Normal</td>
<td>1</td>
<td>1</td>
<td>0.0125</td>
<td>∞</td>
</tr>
<tr>
<td>DF</td>
<td>Decay correction</td>
<td>1.3×10^{-5}</td>
<td>0.0017</td>
<td>Normal</td>
<td>1</td>
<td>2.038</td>
<td>0.003</td>
<td>∞</td>
</tr>
<tr>
<td>( dV/dt )</td>
<td>Voltage reading across the capacitor over the elapsed time</td>
<td>500 fA (ppm)</td>
<td>0.05</td>
<td>Normal</td>
<td>1</td>
<td>1</td>
<td>0.05</td>
<td>∞</td>
</tr>
<tr>
<td>( f_r )</td>
<td>Reproducibility</td>
<td>1</td>
<td>0.1</td>
<td>Normal</td>
<td>1</td>
<td>1</td>
<td>0.1</td>
<td>∞</td>
</tr>
<tr>
<td>BG</td>
<td>Background</td>
<td>0.0015</td>
<td>0.006</td>
<td>Normal</td>
<td>1</td>
<td>2×10^{-4}</td>
<td>1×10^{-6}</td>
<td>∞</td>
</tr>
<tr>
<td>W</td>
<td>Weight of the sample</td>
<td>0.006</td>
<td>0.006</td>
<td>Rectangular</td>
<td>2√3</td>
<td>1</td>
<td>0.02</td>
<td>∞</td>
</tr>
</tbody>
</table>
How to express the answer

Write down the measurement result and the uncertainty, (and record somewhere how you got both of these).

Express the uncertainty as:
 – uncertainty interval
 – coverage factor
 – level of confidence

e.g.

The measured activity was 540 MBq with an expanded uncertainty of ± 2 MBq at a coverage factor $k = 2$ giving a level of confidence of approximately 95 %.
Why use the GUM approach

• Where result “R2” depends on some result “R1” from (say) another lab, the propagation of errors law readily allows the uncertainty in R1 to be incorporated into the uncertainty of R2.

• The combined standard uncertainty can be used to calculate realistic levels of confidence, due to the Central Limit Theorem.