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Coverage Factors for Two-Dimensional Distributions

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COVERAGE FACTORS FOR TWO-DIMENSIONAL DISTRIBUTIONS

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1. INTRODUCTION

In the evaluation of uncertainty related to complex-valued measurements (e.g. scattering parameters as measured with a Vector Network Analyser) coverage factors must be derived in order to account for the probability for the measurand to lie within a given uncertainty region, generally of elliptical form [1], [2], [3], [4], [5]. These coverage factors are dependent on the type of two-dimensional Probability Density Function (PDF) assumed. Traditional approaches assume Bivariate Normal distribution in the real and imaginary components, as well as a coverage factor of approximately 2.45 for a confidence level of 95%.

In real-life measurements, however, magnitude and phase are more meaningful for the performance of the DUT. In RF and microwave measurements, certain parameters such as the Voltage Reflection Coefficient or the Insertion Loss are defined in the form of complex-valued quantities. Specially convenient for computations are the magnitude and phase of these parameters. For measured VRC, the magnitude is directly related to the amount of reflected power from a load, whereas magnitude of IL expressed in dB serves us to easily compute power losses across subsequent components in a microwave circuit. Similarly, the phase of IL is directly related to the electrical delay introduced by a two-port device.

In this Report the analysis of measured data is made in terms of magnitude and phase and compared with the same results as obtained for the real and imaginary components. Also a new model for the PDF is explored, which assumes gaussian distribution for the magnitude and uniform distribution for the phase. This so-called Gaussian Magnitude PDF is compared with the Bivariate Normal distribution in real and imaginary. For both models, coverage factors for the usual confidence level of 95% are derived.

2. CORRELATION OF TWO-DIMENSIONAL DATA

Gererally speaking, correlation can be defined as the degree of mutual relationship between two or more parameters. Let us consider the Real Part and the Imaginary Part of a two-dimensional variable. A Correlation Coefficient tending to zero indicates that the Real and the Imaginary components are independent from each other. In other words: any variation in the *x*-axis is not reflected in a corresponding variation in the *y*-axis, or the Real and Imaginary components are uncorrelated.

On the other hand, a Correlation Coefficient near unity gives us an idea that every change in the *x*-axis tends to be almost exactly reflected as the same variation in the *y*-axis. Positive changes in the *x*-axis correspond themselves with positive changes in the *y*-axis if the Correlation Coefficient is positive, and with negative variations if negative.

The Correlation Coefficient r_{xy} is given by:

$$r_{xy} = \frac{1}{(n-1)\cdot\sigma_x\cdot\sigma_y}\sum_{i=1}^{n}\left(x_i-\bar{x}\right)\cdot\left(y_i-\bar{y}\right) \qquad \text{Eqn. (1)}$$

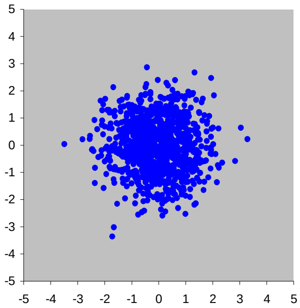
Where $\bar{x}$ and $\bar{y}$ are the Mean Values of the Real and Imaginary components of the set of data, σ_x and σ_y are the Standard Deviations and n is the number of measured data.

The Correlation Coefficient lies within –1 and 1.

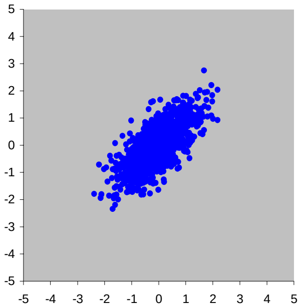
2.1. Some examples of correlation coefficients for two-dimensional data

In the following figures three examples of two-dimensional data sets are shown. Each of them has been generated with 1,000 random trials, and are showing three different Correlation Coefficients. It can be appreciated that, as the absolute value of the Correlation Coefficient approaches unity, the data tend to be distributed along a straight line.

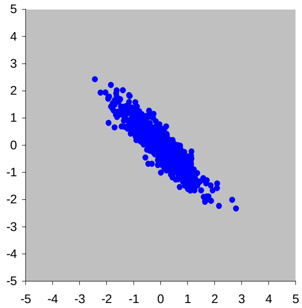
A positive sign in the Correlation Coefficient indicates that every variation in the *x*-axis corresponds itself with a change of equal sign in the *y*-axis. A negative sign in the Correlation Coefficient is an indication that positive changes en the *x*-axis will be seen as negative changes in the *y*-axis, and vice versa.



$r_{xy} = 0.0$



$r_{xy} = 0.7$



$r_{xy} = -0.9$

3. TWO-DIMENSIONAL UNCERTAINTY REGIONS

Uncertainty regions for two-dimensional data are defined, in general, by ellipses. In the particular case where the Correlation Coefficient equals zero, the elliptical uncertainty region is given by:

$$\frac{(x-\bar{x})^2}{\sigma_x^2} + \frac{(y-\bar{y})^2}{\sigma_y^2} = k^2 \qquad \text{Eqn. (2)}$$

It can be easily shown that, if both Standard Deviations are identical, the ellipse becomes a circle of radius $k \cdot \sigma$. Otherwise the above expression is describing an ellipse with no tilt angle. Its mayor axis is extended along the *x*-axis or along the *y*-axis, depending on which Standard Deviation is greater.

In a general case, if the Correlation Coefficient is different from zero, the above expression becomes:

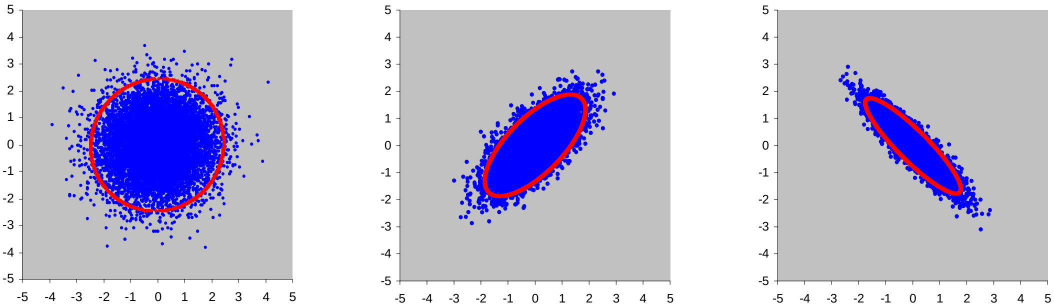
$$\frac{(x-\bar{x})^2}{\sigma_x^2 \cdot \left(1-r_{xy}^2\right)} + \frac{(y-\bar{y})^2}{\sigma_y^2 \cdot \left(1-r_{xy}^2\right)} - \frac{2 \cdot r_{xy} \cdot (x-\bar{x}) \cdot (y-\bar{y})}{\sigma_x \cdot \sigma_y \cdot \left(1-r_{xy}^2\right)} = k^2 \qquad \text{Eqn. (3)}$$

Which describes an ellipse with a non-zero tilt angle. The tilt angle is a function of the correlation between the Real and the Imaginary components, as well as of the Standard Deviations in *x* and *y*. The coverage factor *k* can be adjusted to the desired probability or level of confidence with which it can be stated that the measured value lies within the limits of the ellipse.

3.1. Some examples of two-dimensional uncertainty regions

Here some graphical examples of two-dimensional uncertainty regions are represented. The three data sets have been randomly generated by means of Monte Carlo Simulation, each of them with 10,000 trials. In the first example it can be appreciated that the ellipse has degenerated in a circle.

The coverage factor *k* has been selected so that it can be guaranteed that approximately 95% of occurrences lie within the limits of the uncertainty ellipse. For the two-dimensional distribution chosen (Bivariate Normal) the coverage factor *k* is 2.45 for a confidence level of 95%.



$r_{xy} = 0.0$ $\qquad$ $r_{xy} = 0.7$ $\qquad$ $r_{xy} = -0.9$

4. TWO-DIMENSIONAL PROBABILITY DENSITY FUNCTIONS

When moving from one-dimensional to two-dimensional distributions, the question arises of which Probability Density Function has to be assumed. Let us first introduce some examples of simple one-dimensional PDFs and their two-dimensional counterparts, which will leave us a flavour about the way in which complex-valued quantities are expected to behave.

4.1. The Uniform distribution and its two-dimensional counterpart

The simplest one-dimensional PDF that one can imagine is the Uniform distribution centred around 0. When searching for its counterpart in the complex plane, one tends naturally to apply symmetry of revolution, thus obtaining a nice Cylindrical distribution, uniformly distributed around the coordinate origin within a radius of *a*:

$$PDF(r,\phi) = \frac{1}{\pi \cdot a^2} \quad 0 \le r \le a \qquad\qquad PDF(r,\phi) = 0 \quad elsewhere \qquad \text{Eqn. (4)}$$

Note that we have conveniently chosen the Cylindrical Coordinates *r*, ϕ instead of the Cartesian Coordinate system with the usual real and imaginary components *x*, *y*. The integral over the whole area should satisfy a similar condition as for one-dimensional normalised Probability Density Functions, in this case the volume being unity:

$$\int_{r=0}^{\infty} \int_{\phi=-\pi}^{\pi} PDF(r,\phi) \cdot r \cdot dr \cdot d\phi = 1 \qquad \text{Eqn. (5)}$$

$$\int_{r=0}^{a} \int_{\phi=-\pi}^{\pi} \frac{1}{\pi \cdot a^2} \cdot r \cdot dr \cdot d\phi = \left. \frac{r^2}{2 \cdot \pi \cdot a^2} \right|_{o}^{a} \int_{\phi=-\pi}^{\pi} d\phi = \frac{1}{2 \cdot \pi} \cdot \phi \Big|_{-\pi}^{\pi} = 1 \qquad \text{Eqn. (6)}$$

4.2. The Triangular distribution and its two-dimensional counterpart

Another illustrative, simple one-dimensional PDF is the Triangular distribution, obtained as the convolution of two Uniform distributions. Applying again symmetry of revolution, its two-dimensional counterpart results to be the Cone-shaped distribution, with radius *a* and height $3/(\pi\cdot a^2)$:

$$PDF(r,\phi)=\frac{3}{\pi\cdot a^2}-\frac{3\cdot r}{\pi\cdot a^3} \quad 0\le r\le a \qquad PDF(r,\phi)=0 \quad elsewhere \qquad \text{Eqn. (7)}$$

The volume under the conical surface must also be normalised and thus satisfy the usual condition of Eqn. (5) above:

$$\int_{r=0}^{a}\int_{\phi=-\pi}^{\pi}\left(\frac{3}{\pi\cdot a^2}-\frac{3\cdot r}{\pi\cdot a^3}\right)\cdot r\cdot dr\cdot d\phi=\frac{3}{\pi\cdot a^2}\cdot\left(\frac{r^2}{2}-\frac{r^3}{3\cdot a}\right)\Bigg|_{o}^{a}\int_{\phi=-\pi}^{\pi}d\phi=\frac{1}{2\cdot\pi}\cdot\phi\Big|_{-\pi}^{\pi}=1 \qquad \text{Eqn. (8)}$$

4.3. The Gaussian distribution and its two-dimensional counterpart

If the above process is repeated, one arrives to the one-dimensional Gaussian distribution, which is the result of the convolution of a sufficiently great number of Uniform distributions. Applying symmetry of revolution, the so-called Gaussian Magnitude distribution is obtained, with the magnitude normally distributed and the phase uniformly distributed between $-\pi$ and π:

$$PDF(r,\phi)=\frac{1}{2\cdot\pi\cdot\sigma^2}\cdot\exp\left[\frac{-r^2}{2\cdot\sigma^2}\right] \quad 0\le r\le\infty \qquad \text{Eqn. (9)}$$

Again, the integral over the whole area has to satisfy the usual condition of the resulting volume being unity, as per Eqn. (5):

$$\int_{r=0}^{\infty}\int_{\phi=-\pi}^{\pi}\frac{1}{2\cdot\pi\cdot\sigma^2}\cdot\exp\left[\frac{-r^2}{2\cdot\sigma^2}\right]\cdot r\cdot dr\cdot d\phi=\frac{-1}{2\cdot\pi}\cdot\exp\left[\frac{-r^2}{2\cdot\sigma^2}\right]\Bigg|_{0}^{\infty}\int_{\phi=-\pi}^{\pi}d\phi=\frac{1}{2\cdot\pi}\cdot\phi\Big|_{-\pi}^{\pi}=1 \qquad \text{Eqn. (10)}$$

5. GRAPHICAL REPRESENTATION OF TWO-DIMENSIONAL PDFS

So far we have presented three different two-dimensional distributions for complex-valued quantities which are derived from one-dimensional PDFs: the Uniform (rectangular) PDF nicely transforms itself in a cylinder, whereas the Triangular PDF gives raise to a cone when symmetry of revolution is applied.

This is of course application-dependent. We have intentionally chosen these distributions in the believing that they best represent the expected performance of random and systematic errors for RF measurements. If we instead think of another context, such as the errors encountered in the *X*- and *Y*-deflection of an oscilloscope, we would find these PDFs inadequate. In fact, we would rather move onto a Bivariate Uniform distribution in order to explain deflection errors in the oscilloscope screen. The very definition of the experiment would suggest us to adopt Cartesian Coordinates. In Figures 1 and 2 we represent the two above-mentioned distributions together with their bivariate counterparts for the purpose of comparison between them. They have been generated as two-dimensional histograms by Monte Carlo Simulation (MCS) with a number of trials of 10^6.

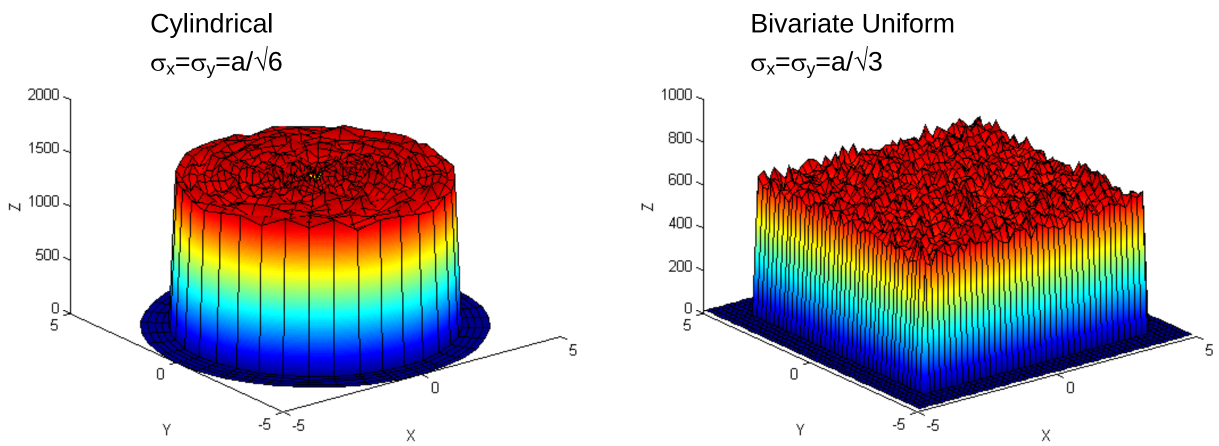


Figure 1. Cylindrical vs. Bivariate Uniform distributions

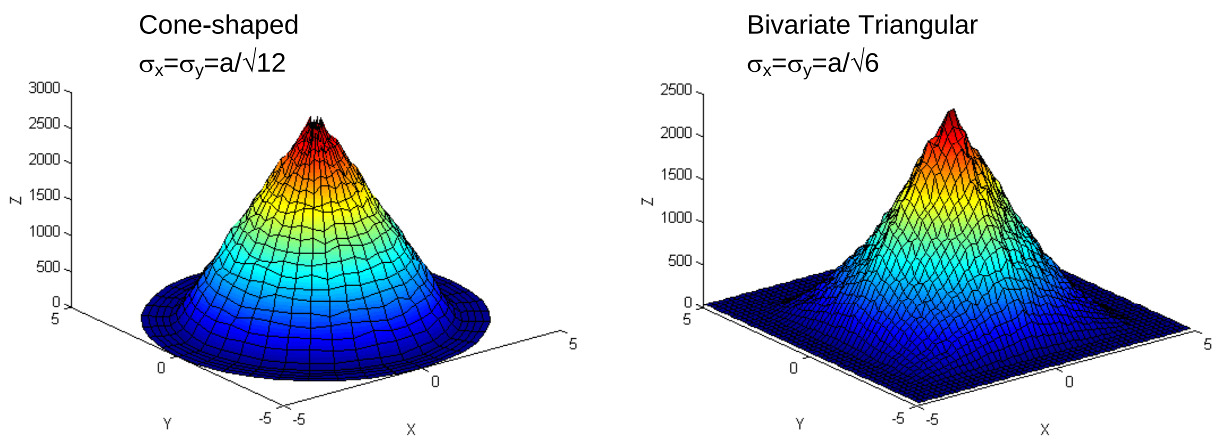


Figure 2. Cone-shaped vs. Bivariate Triangular distributions

In Figures 3 and 4 we represent the PDFs for the proposed Gaussian Magnitude distribution and for the traditional Bivariate Normal, again using MCS with 10^6 trials for the generation of the respective histograms. We have chosen a 2D representation in order to make more evident the (somewhat subtle) differences between them.

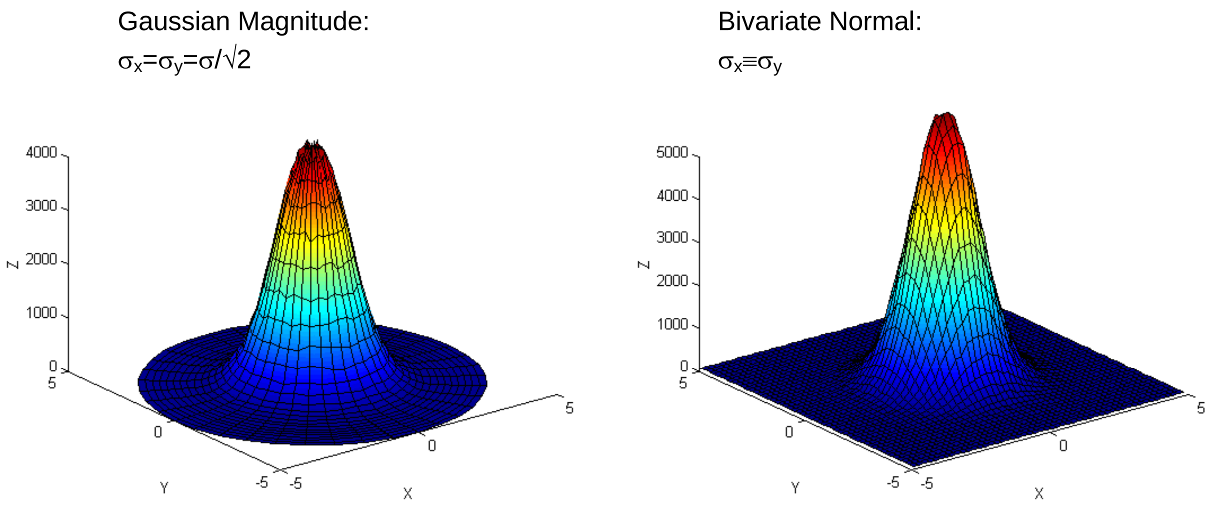


Figure 3. Gaussian Magnitude vs. Bivariate Normal distribution, 3D plot

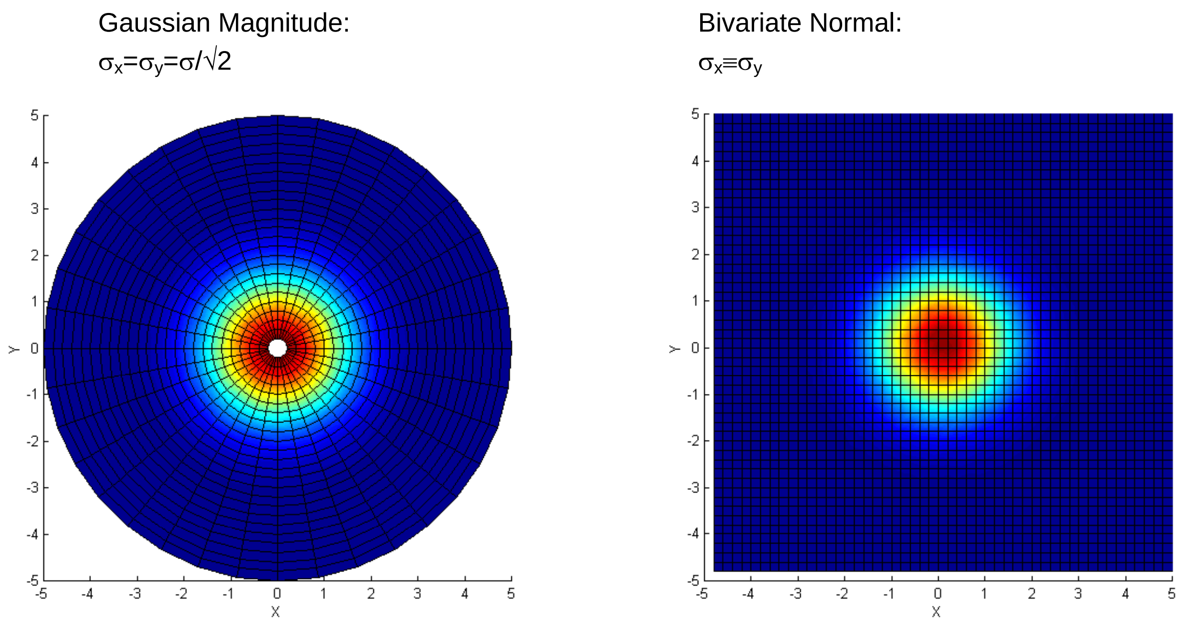


Figure 4. Gaussian Magnitude vs. Bivariate Normal distribution, 2D plot

6. CONVOLUTION OF TWO-DIMENSIONAL PROBABILITY DENSITY FUNCTIONS

The two-dimensional convolution of two functions *f(x,y)* and *g(x,y)* is defined as:

$$h(x, y) = f(x, y) * g(x, y) \equiv \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(z, w) \cdot g(x - z, y - w) \cdot dz \cdot dw \qquad \text{Eqn. (11)}$$

Similarly as for the one-dimensional case, the two-dimensional convolution can be regarded as the result of:

1) Mirroring one of the functions with respect to the coordinate origin, both in *x* and *y*, i.e. *g(z,w)=g(-z,-w)*
2) Gradually moving the previously mirrored function over the non-mirrored one, that is *f(z,w)·g(x-z,y-w)*
3) For each pair *(x,y)* summing up the products obtained from the multiplication of the values of both functions *f(z,w)* and *g(x-z,y-w)* over the whole range of *z* and over the whole range of *w*.

The physical and metrological sense of convolution is as follows. It can be demonstrated that the combination of two independent and uncorrelated two-dimensional random variables, described by their Probability Density Functions *f(x,y)* and *g(x,y)*, can be described by a new Probability Density Function *h(x,y)* which is in turn given by the two-dimensional convolution of *f(x,y)* and *g(x,y)*.

It can be demonstrated that, just as for the one-dimensional case, the Cone-shaped distribution can be obtained as the two-dimensional convolution of two Cylindrical distributions. Also the Gaussian Magnitude distribution is the convolution of a sufficiently great number of Cylindrical distributions, or in general of any kind of Probability Density Functions exhibiting symmetry of revolution.

Similarly, the Bivariate Triangular distribution is the two-dimensional convolution of two Bivariate Uniform distributions, and the Bivariate Normal is the convolution of a great number of Bivariate Uniform distributions, or in general of any kind of bivariate Probability Density Functions.

All the above is applicable to the combination of individual contributions to the uncertainty in measurement. The combination of a sufficiently great number of mutually uncorrelated contributions gives raise to an overall Probability Density Function, this PDF being:

- Gaussian Magnitude, if the individual contributions to the measurement uncertainty are described by Probability Density Functions exhibiting symmetry of revolution.
- Bivariate Normal, in case that the individual uncertainty contributions are better described by bivariate Probability Density Functions.

These observations can be important when predicting the combination of several uncorrelated sources of uncertainty, as it is the case for two-dimensional measurements.

7. THE BIVARIATE NORMAL DISTRIBUTION

Traditionally the Bivariate Normal distribution in real and imaginary components has been assumed for complex-valued quantities. Recalling the previous reasoning, it can be understood that the Bivariate Normal arises as the combination of *n* uncorrelated Bivariate Uniform distributions, just as it may be expected from the interaction among different uncertainty components. However, is this actually the PDF we are expecting to meet in real-life experiments such as measured Reflection Coefficient – or the set of attempts of a skilled archer?

Let us consider the example of the skilled archer, who is able to gaussianly aim at the bull's eye of a target. When aiming, he is supposed to aim first in the *x*-direction, and then, when the aiming in one axis is good enough, fix it and aim in the *y*-direction. This could be roughly describing a bivariate distribution, and if the archer is skilled enough, one may reasonably assume that his attempts will be following a normal or gaussian distribution in *x* and *y*.

Let us represent a set of attempts over the target, together with the PDF exhibited by their magnitude, or distance to the bull's eye. We have made use of MCS with 500 trials for the set of attempts, and 10^6 trials for the magnitude distribution. See Figure 5.

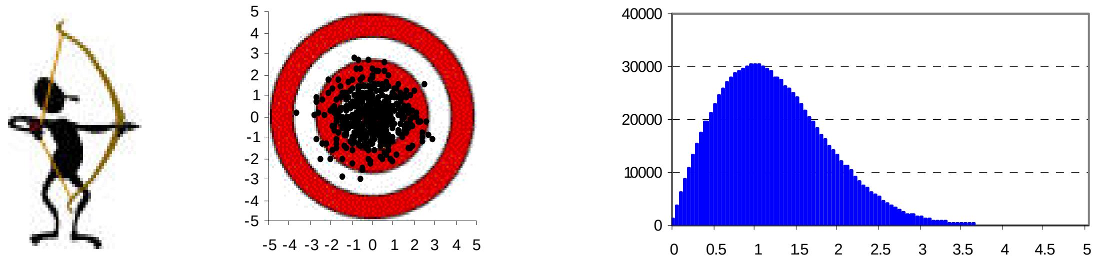


Figure 5. The archer aims gaussianly in *x* and *y*

As mathematics would tell us, the magnitude is following a Rayleigh distribution, with a peak at a value different from the centre of the target. This will probably leave our archer rather upset, since this does not mean the best result possible – taken into account his ability to aim gaussianly at the target.

8. THE GAUSSIAN MAGNITUDE DISTRIBUTION

Let us now try with the Gaussian Magnitude distribution, which can be recalled to have the magnitude normally distributed around 0 and the phase uniformly distributed between $-\pi$ and π. This can be viewed as the combination of *n* uncorrelated Cylindrical distributions, which may be more representative of measured Reflection Coefficient on the Smith Chart. But first let us have a look at our archer.

This time, the archer modifies his strategy and tries to minimise the distance to the bull's eye, paying no attention to the phase. That is, his gaussian ability is now referred to the magnitude. For the purpose of simulation, the phase follows a uniform distribution between $-\pi$ and π.

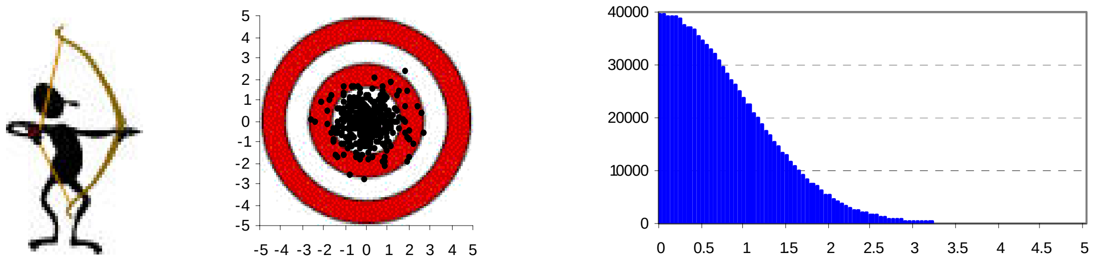


Figure 6. The archer aims gaussianly in magnitude and uniformly in phase

From Figure 6, apart from the fact that the set of hits is better looking, the magnitude distribution seems to be purely gaussian, with a peak in the centre of the target. Even though his gaussian skill aiming in magnitude continues to be as good as aiming in *x* and *y*, the archer happily confirms that he has now achieved a better result! Indeed, his mean score, with a 10 at the bull's eye and a 0 at the edge of the target, has improved from 7.5 to 8.5.

Question: Which model do you think is better suited to describe the performance of a real-life archer?

9. SOME PROPERTIES OF THE GAUSSIAN MAGNITUDE DISTRIBUTION

Bearing in mind the example of the gausssian archer, let us now look at some properties exhibited by the Gaussian Magnitude distribution, such as the Mean Value and Standard Deviation of its magnitude, normalised with respect to the centre value. Note that the determination of the centre (Mean Value of the two-dimensional distribution) continues to be made by means of the usual statistical tools applied to the real and imaginary components.

Without loss of generality, let us assume that the Gaussian Magnitude distribution is centered around the coordinate origin (0, 0).

9.1. Mean Value of the magnitude

In order to determine the Mean Value of its magnitude, first the marginal distribution of the magnitude as a one-dimensional truncated gaussian distribution has to be derived. This half normal distribution for positive values of *r* has the following expression:

$$PDF(r) = \frac{2}{\sqrt{2\cdot\pi}\cdot\sigma}\cdot\exp\left[\frac{-r^2}{2\cdot\sigma^2}\right] \qquad 0 \le r \le \infty \qquad \text{Eqn. (12)}$$

The Mean Value of this marginal distribution is given by integration:

$$\bar{r} = \int_{r=0}^{\infty}\frac{2}{\sqrt{2\cdot\pi}\cdot\sigma}\cdot\exp\left[\frac{-r^2}{2\cdot\sigma^2}\right]\cdot r\cdot dr = -\sqrt{\frac{2}{\pi}}\cdot\sigma\cdot\exp\left[\frac{-r^2}{2\cdot\sigma^2}\right]_0^{\infty} = \sqrt{\frac{2}{\pi}}\cdot\sigma \qquad \text{Eqn. (13)}$$

9.2. Standard Deviation of the magnitude

The Standard Deviation of the magnitude of the above distribution is obtained as the square root of the Variance:

$$\sigma_r^2 = \int_{r=0}^{\infty}\frac{2}{\sqrt{2\cdot\pi}\cdot\sigma}\cdot\exp\left[\frac{-r^2}{2\cdot\sigma^2}\right]\cdot\left(r-\sqrt{\frac{2}{\pi}}\cdot\sigma\right)^2\cdot dr = \ldots$$

$$= \sqrt{\frac{2}{\pi}}\cdot\frac{1}{\sigma}\cdot\left(\exp\left[\frac{-r^2}{2\cdot\sigma^2}\right]\cdot\left(2\cdot\sqrt{\frac{2}{\pi}}\cdot\sigma^3-\sigma^2\cdot r\right)\Bigg|_0^{\infty}+\left(\sqrt{\frac{\pi}{2}}+\sqrt{\frac{2}{\pi}}\right)\cdot\sigma^3\cdot Erf\left[\frac{r}{\sqrt{2}\cdot\sigma}\right]_0^{\infty}\right) = \ldots$$

$$= \sqrt{\frac{2}{\pi}}\cdot\frac{1}{\sigma}\cdot\left(-2\cdot\sqrt{\frac{2}{\pi}}\cdot\sigma^3+\left(\sqrt{\frac{\pi}{2}}+\sqrt{\frac{2}{\pi}}\right)\cdot\sigma^3\right) = \sqrt{\frac{2}{\pi}}\cdot\frac{1}{\sigma}\cdot\left(\sqrt{\frac{\pi}{2}}-\sqrt{\frac{2}{\pi}}\right)\cdot\sigma^3 = \frac{\pi-2}{\pi}\cdot\sigma^2$$

Hence:

$$\sigma_r = \sqrt{\frac{\pi-2}{\pi}}\cdot\sigma \qquad \text{Eqn. (14)}$$

9.3. Scalar coverage factor

In order to obtain the coverage factor which assures to a confidence level *p* that the measurand lies within a given interval, let us integrate the PDF between 0 and a generic radius *a*:

$$p\{r \in [0,a)\} = \int_{r=0}^{a} \frac{2}{\sqrt{2\cdot\pi}\cdot\sigma}\cdot\exp\left[\frac{-r^2}{2\cdot\sigma^2}\right]\cdot dr = Erf\left[\frac{r}{\sqrt{2}\cdot\sigma}\right]_0^a = Erf\left[\frac{a}{\sqrt{2}\cdot\sigma}\right] \qquad \text{Eqn. (15)}$$

Hence, when seeking the coverage factor associated to a confidence level of 95%, the above expression with p=0.95 must be solved for a:

$$Erf\left[\frac{a_{95}}{\sqrt{2}\cdot\sigma}\right] = 0.95 \qquad \text{Eqn. (16)}$$

Which approximately yields:

$$a_{95} = 1.96\cdot\sigma \qquad \text{Eqn. (17)}$$

Which is consistent with the fact that the magnitude is following a true gaussian (although truncated) distribution. Finally, by definition the coverage factor is the ratio of the radius a_{95} to the Standard Deviation of the magnitude:

$$k_{95\,(Scalar)} = \frac{a_{95}}{\sigma_r} = 1.96\cdot\sqrt{\frac{\pi}{\pi-2}} = 3.25 \qquad \text{Eqn. (18)}$$

9.4. Two-dimensional coverage factor

Although the above expressions of Sections 9.1 to 9.3 can help in improving our knowledge of the Gaussian Magnitude distribution, this is not the usual way in which we are going to deal with coverage factors. Recall that this approach assumes that only the magnitude of the distribution, with respect to the Mean Value or centre value, is being computed. Besides, the whole method is not applicable with Correlation Coefficients other than zero. This is why we have denoted $k_{95\,(\mathrm{Scalar})}$ in the previous Section in order to make clear that it takes part of a scalar approach.

However, the results here obtained can be useful when put in relationship with the usual two-dimensional approach in the real and imaginary components (that is the way in which the uncertainty ellipses are obtained). The coverage factor must in this case be defined with respect to the Standard Deviations in *x* and *y*. Note that a_{95} can be also regarded as the radius that encircles 95% of occurrences of a two-dimensional distribution with Correlation Coefficient r_{xy} = 0 and identical Standard Deviations σ_x and σ_y:

$$k_{95\,(2D)} = \frac{a_{95}}{\sigma_x} = \frac{a_{95}}{\sigma_y} = \frac{a_{95}}{\sigma/\sqrt{2}} = 2.77 \qquad \text{Eqn. (19)}$$

This is a two-dimensional coverage factor and as such can be used in Eqns. (2) and (3) for determination of uncertainty circles or ellipses. From now on, mention shall be made only to two-dimensional coverage factors. Recall Section 5 for a experimental demonstration of the relationship between σ, σ_x and σ_y.

Note: The four expressions derived here (Mean Value, Standard Deviation and the two types of coverage factors) have been contrasted with experimental results obtained with Monte Carlo Simulation. Honestly speaking, the experimental results were obtained first (yielding an approximate value of 2.8 for the two-dimensional coverage factor) and subsequently corroborated by mathematics via integration.

10. SOME PROPERTIES OF THE BIVARIATE NORMAL DISTRIBUTION

Just for the sake of comparison between the two distributions, and without empirical demonstration, we quote here the same results obtained for the Bivariate Normal distribution. Again, they have been computed with Monte Carlo Simulation with a number of trials of 10^6. These are approximate values, since they have been obtained by means of numerical tools.

The Mean Value of the magnitude:

$$\bar{r} = 1.25 \cdot \sigma_x = 1.25 \cdot \sigma_y \qquad \text{Eqn. (20)}$$

The Standard Deviation of the magnitude:

$$\sigma_r = 0.655 \cdot \sigma_x = 0.655 \cdot \sigma_y \qquad \text{Eqn. (21)}$$

The radius *a* encircling 95% of occurrences:

$$a_{95} = 2.45 \cdot \sigma_x = 2.45 \cdot \sigma_y \qquad \text{Eqn. (22)}$$

The scalar coverage factor:

$$k_{95\,(Scalar)} = \frac{a_{95}}{\sigma_r} = 3.74 \qquad \text{Eqn. (23)}$$

And finally the two-dimensional coverage factor:

$$k_{95\,(2D)} = \frac{a_{95}}{\sigma_x} = \frac{a_{95}}{\sigma_y} = 2.45 \qquad \text{Eqn. (24)}$$

11. SOME CONSIDERATIONS ABOUT THE CHOICE OF ONE OR THE OTHER DISTRIBUTION (GAUSSIAN MAGNITUDE VS. BIVARIATE NORMAL)

The question whether one or the other distribution has to be chosen to describe an experiment is not easy to respond, and should be made application-dependent. However, in the author's opinion, there are reasons to prefer the Gaussian Magnitude distribution when trying to fix the combined contribution of different sources of uncertainty to complex measured Reflection Coefficient or complex measured Insertion Loss of two-port devices in the frequency range of RF and microwaves. Some of the reasons that might support our choice are:

1) We see no physical reason that may explain the Rayleigh distribution for magnitude of the measured quantity. If it is not expected when measuring a high Reflection Coefficient, why should we be expecting it when the measured VRC tends to the origin, the only difference being a displacement in the coordinate origin?

2) Since the Bivariate Normal distribution is the two-dimensional convolution of non-symmetrical distributions, in principle it should not exhibit symmetry of revolution, as it is the case by definition for the Gaussian Magnitude distribution.

3) We are seeking the combined effect of a series of individual contributions to measurement uncertainty. The Gaussian Magnitude distribution can be explained in terms of the convolution of uncorrelated Cylindrical distributions, whereas the Bivariate Normal would be the result of the two-dimensional convolution of Bivariate Uniform distributions – and these parallelepiped PDFs are unlikely to be met in real-life physical experiments. The Bivariate Normal distribution seems rather to be the unwanted result of the selection of the Cartesian coordinate system when trying to describe an experiment.

Last but not least: Indeed, there are applications which are adequately described by means of bivariate PDFs. Think again of the errors associated to the vertical and horizontal deflection of an oscilloscope screen.

12. COVERAGE FACTORS FOR TWO-DIMENSIONAL DISTRIBUTIONS

Just as for one-dimensional quantities, once the combined uncertainty associated to a measurement has been determined, a coverage factor has to be derived which guarantees with a given probability that the measurand lies within a particular confidence region.

For two-dimensional quantities, the uncertainty regions are ellipses. By means of Monte Carlo Simulation, we can generate a sufficiently great number of random values following any of the above distributions. Subsequently, we compute the proportion (in percentage) of occurrences that are met within different elliptical regions. In Table 1 we are showing the associated confidence levels obtained for different coverage factors in a similar format as in [1].

Coverage factor	Confidence level		
	Scalar measurement	Vector measurement (Bivariate Normal)	Vector measurement (Gaussian Magnitude)
k=1	68.3%	39.3%	52.0%
k=2	95.4%	86.5%	84.3%
k=2.45	98.6%	95.0%	91.7%
k=2.77	99.4%	97.8%	95.0%

Table 1. Comparison between coverage factors for scalar and complex-measured quantities

As it can be seen, a coverage factor k_{BN} = 2.45 guarantees a confidence level of 95% for a Bivariate Normal distribution, although it might not be adequate for a Gaussian Magnitude distribution. For the latter, and for the same confidence level, a coverage factor k_{GM} = 2.77 is proposed, according to the results obtained in Section 9.4.

Apparently, a greater coverage factor is consistent with the fact (observed experimentally) that the Gaussian Magnitude distribution is more concentrated around the Mean Value. This observation has to be clarified, recalling again the two examples of Figures 2 and 3.

For the Bivariate Normal we have made the Standard Deviations in the real and imaginary components, $\sigma_x = \sigma_y = 1$ identical to the Standard Deviation $\sigma = 1$ for the Gaussian Magnitude distribution, in the believing that this allows us to compare them in identical conditions. The radius of the circle encircling 95% of occurrences would be given by:

Gaussian Magnitude: $$a_{95} = k_{GM} \cdot \frac{\sigma}{\sqrt{2}} = 1.96$$ Eqn. (25)

Bivariate Normal: $$a_{95} = k_{BN} \cdot \sigma_x = k_{BN} \cdot \sigma_y = 2.45$$ Eqn. (26)

This corroborates our initial observation that the Gaussian Magnitude distribution is more concentrated around the Mean Value. In order to do the same check for any confidence level, empirical formulas are needed which relate confidence levels and coverage factors. Once these formulas are obtained, the MCS method is no longer needed for determination of generic coverage factors in whichever application.

12.1. Empirical formulas for coverage factors

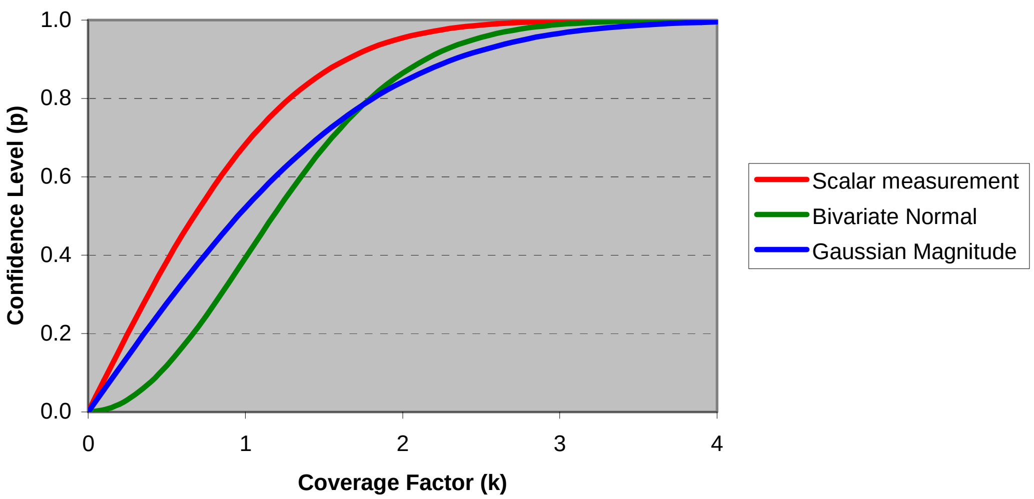


Figure 7. Confidence level as a function of coverage factor

In Figure 7 we represent the relationship between coverage factor and confidence level. The three curves shown are given by the following expressions:

Scalar measurement: $$p = Erf\left(\frac{k}{\sqrt{2}}\right) = 1 - t\,Distribution\left(k; \nu_{eff} = \infty\right)$$ Eqn. (27)

Bivariate Normal: $$p = 1 - Exp\left(\frac{-k^2}{2}\right)$$ Eqn. (28)

Gaussian Magnitude: $$p = Erf\left(\frac{k}{2}\right) = 1 - t\ Distribution\left(\frac{k}{\sqrt{2}}; \nu_{eff} = \infty\right)$$ Eqn. (29)

The Student´s *t*-Distribution has been introduced in order to possibly account for a number of effective degrees of freedom other than infinity. Its use, though, has not been demonstrated yet, neither experimentally nor empirically.

The inverse *t*-Distribution allows us to obtain also the coverage factor as a function of the confidence level:

Scalar measurement: $$k = Inverse\ t\ Distribution(1 - p; \nu_{eff} = \infty)$$ Eqn. (30)

Bivariate Normal: $$k = \sqrt{-2 \cdot Ln(1 - p)}$$ Eqn. (31)

Gaussian Magnitude: $$k = \sqrt{2} \cdot Inverse\ t\ Distribution(1 - p; \nu_{eff} = \infty)$$ Eqn. (32)

13. CONCLUSIONS

We have presented here some results of the MCS method applied to the generation of two-dimensional PDFs. In particular, an alternative distribution has been explored, namely the Gaussian Magnitude (gaussian in magnitude and uniform in phase), and compared to the traditional Bivariate Normal. In the light of the results obtained, new coverage factors have been proposed, which take into account the expected distribution of measured data around the Mean Value.

14. REFERENCES

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