ANAMET Report 037
November 2002

An Approach to the Treatment of Uncertainty in Complex S-Parameter Measurement

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This report has been approved by the ANAMET Steering Committee

AN APPROACH TO THE TREATMENT OF UNCERTAINTY
IN COMPLEX *S*-PARAMETER MEASUREMENTS

Foreword

This report contains a re-print of an article that appeared recently in Metrologia – an international journal dealing with the scientific aspects of metrology (www.bipm.fr/metrologia) published by the Bureau International des Poids et Mesures (BIPM).

The article describes a programme of work, undertaken by the UK's National Physical Laboratory, aimed at developing reliable methods for dealing with the uncertainty of measurements made using Vector Network Analysers (VNAs). The work forms part of the 2000–2003 Electromagnetic Metrology Programme funded by the National Measurement System Policy Unit of the UK government's Department of Trade and Industry (www.dti.gov.uk/nmd).

Although the article was published originally in Metrologia, it is reproduced here due to its potential interest to the ANAMET community.

Nick Ridler
Chairman of ANAMET

October 2002

An approach to the treatment of uncertainty in complex S-parameter measurements

N. M. Ridler and M. J. Salter

Abstract. This paper presents methods for evaluating, expressing and using the uncertainty associated with complex S-parameter measurements. The methods are based on internationally recommended guidelines, published by the International Organization for Standardization (ISO), with extensions to accommodate the complex nature of the measurands. The treatment of measurements from both one-port and multi-port devices is presented and this is used to propagate uncertainty from complex-valued S-parameters to radio frequency (rf) and microwave quantities derived from these. A simple example, involving the comparison loss correction used during a microwave power meter calibration, is included to demonstrate the principles of propagating the uncertainty in complex S-parameter measurements.

1. Introduction

A current ISO guidance document [1] gives detailed information on evaluating and expressing the uncertainty in measurement results. However, the emphasis in this document is on dealing with scalar measurement quantities.[1] Therefore, this information is of limited use when dealing with vector measurement quantities, such as complex reflection and transmission coefficients (i.e. S-parameters) encountered in rf and microwave science and technology.

Several important issues need to be addressed when dealing with complex-valued measurement data, such as the form to be used to express the complex quantity (in terms of real and imaginary components or magnitude and phase), and any correlation between these components. This is particularly true when a detailed description of the uncertainty in measurement of the complex quantity is required, including the need to apply suitable coverage factors to obtain uncertainty statements at a specific level of confidence.

This paper addresses these issues and presents what is effectively an extension of the principles given in the ISO *Guide* [1] appropriate for complex-valued quantities. Consideration is given to S-parameter measurements of both one-port and multi-port devices and to how such measurements can be used to evaluate some other output quantity of interest determined from the S-parameter data, together with its corresponding uncertainty. This is illustrated using a simple example from microwave power meter calibration.

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2. Parameter representations

The alternative ways of representing a complex measurand are in terms either of its real and imaginary components or its magnitude and phase. In the majority of engineering applications, the magnitude and phase representation is generally preferred because this representation bears a direct relationship to physical phenomena affecting the measurement process. For example, phase relates directly to the electrical path length of a signal, and magnitude relates directly to the signal attenuation. The same cannot be said for the representation of a complex measurand in terms of its real and imaginary components.

However, when evaluating the uncertainty in a complex measurand, additional considerations need to be taken into account. These relate mainly to the use of statistics to evaluate the random errors in the measurement process. In particular, applying statistics to analyse data involves performing a number of arithmetic operations on the data (e.g. addition, subtraction, etc.). If the scales used to depict the different representations of a complex measurand are examined, the following is found: the real and imaginary axes in the complex plane extend indefinitely, in principle, in both directions (i.e. to $\pm\infty$). This is the same as the scale used to depict all real numbers, which are routinely subject to the arithmetic operations mentioned above. However, the scales used to represent magnitude and phase each possess a significant difference. The magnitude has a lower bound of zero below which values cannot exist, and phase is

1. See, for example, § 3.1.7 of [1], where it states that "This *Guide* treats the measurand as a scalar (a single quantity)".

represented conventionally on a cyclical scale, usually either from –180° to +180° or from 0° to 360°.

The characteristics of both these scales can cause problems when using statistics to summarize complex-valued data. For example, if repeat measurements are made of a low-magnitude S-parameter, a systematic error is introduced if the magnitude values are averaged. This method of averaging can be expressed mathematically as

$$\overline{|S|} = \frac{1}{n}\sum_{i=1}^{n} |S_i|, \tag{1}$$

where $\overline{|S|}$ is the mean magnitude of an arbitrary S-parameter and $|S_i|$ represents the magnitude of the i-th of n measurements of S. Now, for a perfect matched-load termination, $|S_{11}| = 0$. However, every measurement of $|S_{11}|$ will always be greater than zero due to random errors in the measurement process. If the repeat measurements of S_{11} (complex) were displayed in the complex plane, it can be assumed that they would appear as a two-dimensional, i.e. bivariate, normal distribution centred on the origin of the complex plane. However, when the distribution of $|S_{11}|$ is plotted, instead of the usual situation of a normal distribution centred on the notionally "true" value[2] (i.e. $|S_{11}| = 0$), we find that the distribution is skewed (i.e. a Rayleigh distribution)[3] with the peak at a value greater than the "true" value, $|S_{\mathbf{true}}|$. Hence, $\overline{|S|} > |S_{\mathbf{true}}|$ when $|S_{\mathbf{true}}|$ is small, i.e. $\overline{|S|}$ becomes a biased estimator of $|S_{\mathbf{true}}|$.

A similar problem occurs when performing a Type A evaluation of the standard uncertainty in $\overline{|S|}$, $u(\overline{|S|})$, using

$$u(\overline{|S|}) = \sqrt{\frac{1}{n(n-1)}\sum_{i=1}^{n}(|S_i| - \overline{|S|})^2}. \tag{2}$$

In this case, the calculation of the standard uncertainty underestimates the "true" underlying standard uncertainty. Both of these effects are discussed in detail in [2].

Calculations performed on phase measurements also present difficulties for conventional arithmetic. The cyclical nature of the phase scale requires a form of "clock" (or modular) arithmetic to be used when performing calculations. For example, averaging the phase values +179° and –177° using conventional arithmetic leads to a result of +1°, as

$$\frac{(+179^\circ) + (-177^\circ)}{2} = +1^\circ,$$

whereas, intuitively, one would expect the result to be –178° (on a scale ranging from –180° to +180°).

Considering the above difficulties, it is recommended that complex S-parameter measurements should be analysed in terms of real and imaginary components, resulting in the following calculations for a given complex S-parameter:

(a) Calculate the mean of the real component, $\overline{\mathrm{Re}(S)}$, using

$$\overline{\mathrm{Re}(S)} = \frac{1}{n}\sum_{i=1}^{n}\mathrm{Re}(S_i), \tag{3}$$

where $\mathrm{Re}(S_i)$ denotes the real component of the i-th of n measurements of S.

(b) Calculate the mean of the imaginary component, $\overline{\mathrm{Im}(S)}$, using

$$\overline{\mathrm{Im}(S)} = \frac{1}{n}\sum_{i=1}^{n}\mathrm{Im}(S_i), \tag{4}$$

where $\mathrm{Im}(S_i)$ denotes the imaginary component of the i-th measurement of S.

(c) Calculate the standard uncertainty in the mean of the real component, $u[\mathrm{Re}(S)]$, using

$$u[\mathrm{Re}(S)] = \sqrt{\frac{1}{n(n-1)}\sum_{i=1}^{n}[\mathrm{Re}(S_i) - \overline{\mathrm{Re}(S)}]^2}. \tag{5}$$

(d) Calculate the standard uncertainty in the mean of the imaginary component, $u[\mathrm{Im}(S)]$, using

$$u[\mathrm{Im}(S)] = \sqrt{\frac{1}{n(n-1)}\sum_{i=1}^{n}[\mathrm{Im}(S_i) - \overline{\mathrm{Im}(S)}]^2}. \tag{6}$$

It is *not* recommended that (1) and (2) are used to summarize the magnitude of complex-valued data. Instead, if required, calculated values should be transformed into a polar representation (i.e. magnitude and phase) only after all computations have been performed on the data.

2. Ignoring any effects caused by systematic errors in the measurement process.

3. An analogy here is when a skilled darts player (i.e. one with a good aim, and without any systematic bias) throws darts at a dartboard aiming at the point centre (a point with zero area). If the distance from the point centre to the position where each dart has landed is measured, and the frequency of these distances is plotted, the same distribution is produced (which is also truncated at zero).

3. Correlation and the covariance matrix

Having shown it to be more appropriate to analyse complex-valued data in terms of real and imaginary components, consideration needs to be given to whether there is any correlation between the components. Correlation can be described as the degree to which variations in the two components are interrelated. For example, if a variation in some physical process (such as the misalignment of a microwave connection) causes both real and imaginary components of a given S-parameter to increase, then these components are said to be positively correlated. If, on the other hand, variation in the physical process causes one component to increase while the other component decreases, then this is termed negative correlation.

The degree of correlation between the two components of S, i.e. $\mathrm{Re}(S)$ and $\mathrm{Im}(S)$, can be characterized by the estimated correlation coefficient, $r[\mathrm{Re}(S),\mathrm{Im}(S)]$, which is calculated as follows:

$$r[\mathrm{Re}(S),\mathrm{Im}(S)] = \frac{u[\mathrm{Re}(S),\mathrm{Im}(S)]}{u[\mathrm{Re}(S)]\,u[\mathrm{Im}(S)]}, \tag{7}$$

where $u[\mathrm{Re}(S),\mathrm{Im}(S)]$ is the estimated covariance in the means of $\mathrm{Re}(S)$ and $\mathrm{Im}(S)$, and is given by

$$u[\mathrm{Re}(S),\mathrm{Im}(S)] = \frac{1}{n(n-1)}\sum_{i=1}^{n}[\mathrm{Re}(S_i)-\overline{\mathrm{Re}(S)}][\mathrm{Im}(S_i)-\overline{\mathrm{Im}(S)}]. \tag{8}$$

An alternative way of expressing this information is using the covariance matrix

$$\begin{pmatrix} u^2[\mathrm{Re}(S)] & u[\mathrm{Re}(S),\mathrm{Im}(S)] \\ u[\mathrm{Im}(S),\mathrm{Re}(S)] & u^2[\mathrm{Im}(S)] \end{pmatrix}, \tag{9}$$

where $u[\mathrm{Re}(S),\mathrm{Im}(S)] = u[\mathrm{Im}(S),\mathrm{Re}(S)]$, as the covariance matrix is symmetric.

The uncertainty in a complex measurand can therefore be expressed in two ways: either (i) using the three terms: uncertainty in the real component; uncertainty in the imaginary component; and correlation coefficient; or (ii) as the covariance matrix.

Example

The following example shows how to evaluate the result of a measurement of a complex measurand, along with its uncertainty, using the techniques presented above. For simplicity, this example assumes that only random errors affect the measurement process and that these can be evaluated using statistical techniques.

Table 1 shows six repeat measurements of S_{11} of a device at an arbitrary frequency and Table 2 shows the summary statistics obtained from applying (3) to (7) to these data.

Table 1. Six repeat S_{11} measurements of a device at an arbitrary frequency.

Measurement number	Real component of S_{11}/linear units	Imaginary component of S_{11}/linear units
1	0.1847	0.1866
2	0.1852	0.1924
3	0.2072	0.1925
4	0.2003	0.1880
5	0.2031	0.2080
6	0.2044	0.2233

Table 2. Summary statistics obtained from the measurements presented in Table 1.

Summary statistic	Real component of S_{11}/linear units	Imaginary component of S_{11}/linear units
Mean	0.1975	0.1985
Standard uncertainty in the mean	0.0041	0.0059
Correlation coefficient[4]	+0.5	

Alternatively, the summary information given in Table 2 can be presented in terms of the means in the real and imaginary components of S_{11}, along with the elements of the covariance matrix, as follows:

$$S_{11} = 0.1975 + \mathrm{j}\,0.1985, \text{ with covariance matrix}$$
$$\begin{pmatrix} 16.541\times10^{-6} & 12.178\times10^{-6} \\ 12.178\times10^{-6} & 34.347\times10^{-6} \end{pmatrix}.$$

4. Geometric representation of uncertainty

Another aspect of the uncertainty in measurement of a complex-valued quantity is its geometric representation. In the case of a scalar measurand, the uncertainty can be thought of as an interval, in one-dimensional space, extending on either side of the mean measured value. However, for a complex-valued (vector) measurand, it is more appropriate to think in terms of a two-dimensional *region of uncertainty* in the complex plane. Based on the assumption that the data emanate from a bivariate normal distribution, the ellipse defined by the positive definite covariance matrix and centred on the complex

4. It is rarely meaningful to express an estimated correlation coefficient to more than one significant figure. This is because, for most practical measurement situations, such estimates are usually based on relatively small sample sizes and are hence inherently unreliable (as shown in Section 5).

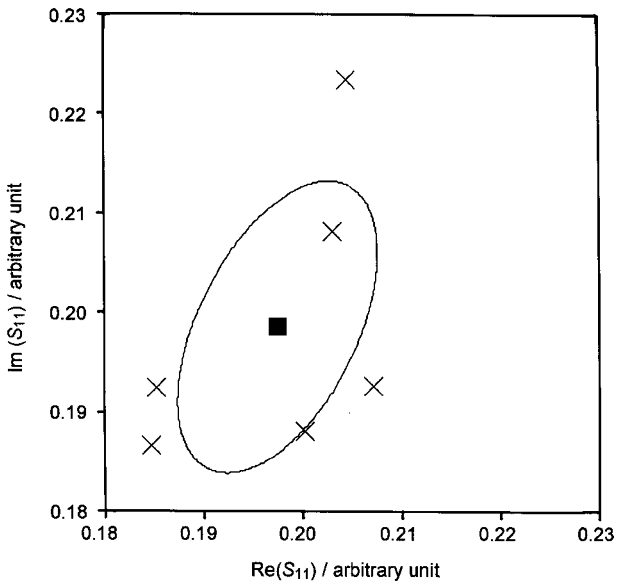

Figure 1. Elliptical region of uncertainty (95 % level of confidence) for the repeat measurements considered in the example given in the text. Measured values are shown as crosses and the mean value is shown as a square.

mean value defines such a region. The size, aspect ratio and orientation with respect to the coordinate axes for the ellipse are governed by the elements of the covariance matrix (or the uncertainty in the real and imaginary components, along with the correlation coefficient).

To illustrate this, Figure 1 shows the ellipse of uncertainty obtained from the example considered earlier. This figure shows the six repeat measurements as crosses in the plane, along with a square showing the mean value, and the uncertainty ellipse centred on the mean value. The ellipse has been scaled to provide a region of uncertainty at a 95 % level of confidence (see Section 7).

A geometric representation of a complex-valued uncertainty can be very useful in observing and understanding trends in S-parameter uncertainties. For example, changes in the size, shape and orientation of the uncertainty ellipse can be examined as a function of the measurement frequency. Such an investigation has been presented elsewhere [3].

5. Reliability of a sample correlation coefficient

Having introduced the correlation coefficient as a measure of the interdependence between the real and imaginary components of a complex measurand, it is worth considering the reliability of an estimated correlation coefficient based on a given number of repeat measurements regarded as a sample drawn from the population of all possible values.

The reliability of an estimate of a correlation coefficient can be established by evaluating the uncertainty, at a given level of confidence, in the estimate. This is similar to an uncertainty assigned to any statistic, such as the sample mean. However, the usual rules for establishing such an interval rely on normal distribution theory, so cannot be applied here as the distribution of the correlation coefficient is often very non-normal and asymmetric. This is because the correlation coefficient is a bounded quantity[5] having an upper bound of +1 and a lower bound of –1, i.e.

$$-1 \le r \le +1.$$

A detailed discussion on the distribution of an estimated correlation coefficient is given in [4].

This situation is dealt with by applying the Fisher z-transformation [5], which converts any distribution in r into a near-normal distribution in z. Therefore, normal distribution theory can be applied to the z-statistic and uncertainty intervals obtained that can then be transformed back to the equivalent uncertainty intervals in r.

The Fisher z-transformation is given by

$$z = \frac{1}{2}\log_e \frac{1+r}{1-r}. \qquad (10)$$

It follows that

$$r = \tanh z. \qquad (11)$$

The standard deviation in z, $s(z)$, is given approximately by

$$s(z) = \frac{1}{\sqrt{n-3}}, \qquad (12)$$

where n is the number of measurements used to estimate the correlation coefficient. An expanded uncertainty, U, in z can be calculated, in the usual way:

$$U = k\, s(z), \qquad (13)$$

where k is a coverage factor chosen to provide the required level of confidence. For example, $k \approx 2$ provides a level of confidence of 95 %.

Therefore, an uncertainty, at a given level of confidence, can be obtained for an estimated correlation coefficient, as follows.

(a) Convert r to z using (10).

5. The boundaries (beyond which a value cannot exist) on the correlation coefficient scale cause a skewing to the distribution of values.

(b) Calculate $s(z)$ using (12).

(c) Calculate U using (13).

(d) Use z and U to establish the maximum and minimum values of z, $z_{\max}$ and $z_{\min}$ respectively, due to U, i.e. $z_{\max} = z + U$ and $z_{\min} = z - U$.

(e) Convert $z_{\max}$ and $z_{\min}$ to the equivalent $r_{\max}$ and $r_{\min}$ using (11).

(f) Use r, $r_{\max}$ and $r_{\min}$ to obtain the uncertainty in r, i.e. the upper interval is given by $r_{\max} - r$, and the lower interval by $r - r_{\min}$. It is necessary to calculate the upper and lower intervals of the uncertainty separately as the overall uncertainty interval is likely to be asymmetric.

Applying this procedure to the example given earlier gives the following result:

$$r = +0.5\left\{\pm\begin{matrix}0.4\\1.0\end{matrix}\right\},$$

where the uncertainty interval is quoted at a 95 % level of confidence. Clearly, this uncertainty interval is both asymmetric and relatively large (remembering that the total range of values for r extends from –1 to +1). To obtain a more reliable estimate of r, more repeat measurements are needed [as indicated by (12)].

6. Extension to multi-port devices

The above approach to evaluating and expressing the uncertainty in a complex S-parameter measurement can be extended to deal with multiple S-parameter measurements made simultaneously on multi-port, or n-port, devices. For each real and imaginary component of each S-parameter, the mean and standard uncertainty in the mean is calculated, along with the correlation coefficient for each real and imaginary pair. However, in addition to this, correlation *between* the S-parameters (both real and imaginary components) needs to be considered.

For example, for a two-port device, four complex S-parameters are measured. This results in an (8 × 8) covariance matrix, containing 28 off-diagonal elements, resulting in 28 correlation coefficients representing all possible combinations of pairs of S-parameter components. Some of the elements of this covariance matrix are shown below (assuming that the S-parameters are arranged as S_{11}, S_{21}, S_{12} and S_{22}), using dots to indicate the position of the missing elements.

$$\begin{pmatrix}
u^2[\mathrm{Re}(S_{11})] & u[\mathrm{Re}(S_{11}),\mathrm{Im}(S_{11})] & . & \ldots & u[\mathrm{Re}(S_{11}),\mathrm{Im}(S_{22})] \\
u[\mathrm{Im}(S_{11}),\mathrm{Re}(S_{11})] & u^2[\mathrm{Im}(S_{11})] & . & \ldots & u[\mathrm{Im}(S_{11}),\mathrm{Im}(S_{22})] \\
. & . & u^2[\mathrm{Re}(S_{21})] & \ldots & . \\
. & . & . & \ldots & . \\
. & . & . & \ldots & . \\
. & . & . & \ldots & . \\
u[\mathrm{Re}(S_{22}),\mathrm{Re}(S_{11})] & . & . & \ldots & u[\mathrm{Re}(S_{22}),\mathrm{Im}(S_{22})] \\
u[\mathrm{Im}(S_{22}),\mathrm{Re}(S_{11})] & . & . & \ldots & u^2[\mathrm{Im}(S_{22})]
\end{pmatrix}$$

In general, to fully characterize an n-port device, there will be n^2 S-parameter measurements, with a $(2n^2 \times 2n^2)$ covariance matrix, resulting in $n^2(2n^2-1)$ correlation coefficients. Table 3 gives values for these characteristics for complex one-, two-, three- and four-port device measurements. In terms of providing a geometric representation of the uncertainty, this would require displaying an uncertainty ellipsoid in $2n^2$-dimensional space.

Table 3. Numbers of characteristics required for n-port devices $(n = 1, 2, 3, 4)$.

Number of device ports	Number of complex S parameters	Size of covariance matrix	Number of correlation coefficients
1	1	(2 × 2)	1
2	4	(8 × 8)	28
3	9	(18 × 18)	153
4	16	(32 × 32)	496

7. Coverage factors

Having established the standard uncertainties for all the components in the complex-valued measurands, it is conventional to apply a coverage factor to obtain an expanded uncertainty at a required level of confidence. For example, for scalar quantities a coverage factor of $k = 1.96$ (usually approximated to $k = 2$) can be used to establish an expanded uncertainty at a 95 % level of confidence.[6] This is obtained as the 97.5 % point of the standard normal distribution. If the number of degrees of freedom, ν, is finite then the corresponding coverage factor is given by the 97.5 % point of Student's t-distribution with ν degrees of freedom. For example, for $\nu = 5$, then $k = 2.57$ [1].

However, for multiple complex-valued measurands, a coverage factor of $k = 2$ will no longer yield an expanded uncertainty at a level of confidence of 95 % (even when the number of degrees of freedom is infinite). For example, a value of approximately $k = 2.45$ is required to provide a level of confidence of 95 % for a single complex S-parameter. This coverage factor k is obtained as the square root of the 95 % point of the χ^2-distribution with two degrees of freedom as

6. Assuming that the underlying distribution characterizing the measurand is normal and that the uncertainty is known with a sufficiently high number of degrees of freedom.

a degree of freedom is required for each component of the complex measurand. In general, for an n-port device for which n^2 S-parameters are measured, the coverage factor for a level of confidence of 95 % is given by

$$k = \sqrt{\chi^2_{p,0.95}},$$

where $p = 2n^2$ and $\chi^2_{p,0.95}$ is the 95 % point of the χ^2-distribution with p degrees of freedom [6]. This allows coverage factors for n-port measurements to be derived. These are given in Table 4, for complex one-, two-, three- and four-port measurements. If the number of degrees of freedom, ν, is finite then the corresponding coverage factor is given by [6]

$$k = \sqrt{\frac{\nu p}{\nu + 1 - p} F_{p,\nu+1-p,0.95}},$$

where $F_{p,\nu+1-p,0.95}$ is the 95 % point of the F-distribution whose degrees of freedom parameters are equal to p and $\nu + 1 - p$. Table 5 lists some coverage factors for a single complex S-parameter for different numbers of degrees of freedom.

Table 4. Coverage factors required to give an expanded uncertainty with a level of confidence of 95 % for multi-port complex S-parameter measurements.

Number of device ports	k factor
1	2.45
2	3.94
3	5.37
4	6.80

Table 5. Coverage factors required to produce an expanded uncertainty with a level of confidence of 95 % for one-port complex S-parameter measurements with a finite number of degrees of freedom.

Degrees of freedom	k factor
2	28.26
3	7.55
4	5.05
5	4.17
6	3.73
7	3.46
8	3.29
9	3.17
10	3.08
100	2.50
1000	2.45
∞	2.45

8. Using complex-valued measurement results

Complex S-parameter measurements are often used to determine some other quantity of interest. In effect, the S-parameter measurements become input quantities to a subsequent measurement process, resulting in an output quantity (or quantities). Generalized measurement system models dealing with such relationships, including complex-valued quantities, are presented in [7]. Such models can be applied to propagate the uncertainties in the input quantities to those in the output quantities. If n input S-parameters, $S_i (i = 1, ..., n)$, are related to m (assumed here also to be complex) output quantities, ξ_j $(j = 1, ..., m)$, by a functional relationship, f, then this can be expressed as

$$\xi = f(S). \tag{14}$$

The uncertainty in ξ is represented by its $(2m \times 2m)$ covariance matrix, $V(\xi)$:

$$V(\xi) = JV(S)J^{\mathrm{T}}, \tag{15}$$

where $V(S)$ is the $(2n \times 2n)$ covariance matrix for the n input S-parameters, and J is the $(2m \times 2n)$ Jacobian matrix of f with respect to S. All information concerning the uncertainty components and correlation coefficients in the output quantities is contained within the elements of $V(\xi)$, and hence this is an alternative form of the law of propagation of uncertainty given in [1].

This approach can be applied to all situations where the results of complex S-parameter measurements are used to establish some other measurement quantity. The following example illustrates this approach for a comparison loss measurement made during calibration of a power meter. In this example, ξ consists of a single scalar output quantity and its covariance matrix, $V(\xi)$, is (1×1). Only a special case of comparison loss measurement is considered here, as this case demonstrates all the salient features with this approach to propagating uncertainties in complex S-parameters. More general cases of propagating S-parameter uncertainties in comparison loss measurements are dealt with in detail in [8].

9. Comparison loss example

During the calibration of a microwave power meter, the power meter under test and a standard power meter are connected in turn to a stable signal generator. The power absorbed by the two power meters will be different, in general, because their voltage reflection coefficients (VRCs) have different complex values. The ratio, M, of the power absorbed by the power meter under test (P_{M}) to the power absorbed by the standard power meter (P_{S}) is given by

$$M = \frac{P_{\mathrm{M}}}{P_{\mathrm{S}}} = \frac{1 - |\Gamma_{\mathrm{M}}|^2}{1 - |\Gamma_{\mathrm{S}}|^2} \frac{|1 - \Gamma_{\mathrm{S}}\Gamma_{\mathrm{G}}|^2}{|1 - \Gamma_{\mathrm{M}}\Gamma_{\mathrm{G}}|^2}, \tag{16}$$

where Γ_{G} is the VRC of the signal generator, Γ_{M} is the VRC of the power meter under test and Γ_{S} is the VRC of the standard power meter. (All VRCs are complex quantities and are equivalent to S-parameters with equal indices, i.e. S_{11}, S_{22}, etc.) The power ratio in (16) is often referred to as the "comparison loss" [9].

A special, yet important, case of (16) occurs when the standard power meter and the signal generator are assumed to be reflectionless (i.e. $\Gamma_S = \Gamma_G = 0$). Under these circumstances, the comparison loss becomes

$$M = 1 - |\Gamma_M|^2. \tag{17}$$

This can be written in terms of the real and imaginary components of the VRC of the power meter under test as follows:

$$M = 1 - x^2 - y^2,$$

where $x = \mathrm{Re}(\Gamma_M)$ and $y = \mathrm{Im}(\Gamma_M)$. In this case, the two-dimensional vector input quantity is (x, y) and the one-dimensional (i.e. scalar) output quantity is simply M. The (2×2) covariance matrix for the vector input quantity is

$$V(\Gamma_M) = \begin{bmatrix} u^2(x) & u(x,y) \\ u(y,x) & u^2(y) \end{bmatrix}$$

and the (1×1) covariance matrix of the output quantity is $V(M) = \lfloor u^2(M) \rfloor$. The (1×2) Jacobian matrix of the transformation is given by

$$J = \left[\frac{\partial M}{\partial x}\frac{\partial M}{\partial y}\right] = [-2x \;\; -2y].$$

Now, by the law of propagation of uncertainty [i.e. (15)],

$$V(M) = JV(\Gamma_M)J^{\mathrm{T}}$$

$$[u^2(M)] = [-2x \;\; -2y]\begin{bmatrix} u^2(x) & u(x,y) \\ u(y,x) & u^2(y) \end{bmatrix}\begin{bmatrix} -2x \\ -2y \end{bmatrix}.$$

Multiplying out the matrices and using the fact that $u(x, y) = u(y, x)$ gives the uncertainty in M as

$$u(M) = 2[u^2(x)x^2 + u^2(y)y^2 + 2u(x,y)xy]^{1/2}$$

or, in terms of the correlation coefficient, $r(x, y)$, between real and imaginary components of Γ_M:

$$u(M) = 2[u^2(x)x^2 + u^2(y)y^2 + 2r(x,y)u(x)u(y)xy]^{1/2}. \tag{18}$$

Equation (18) shows that the uncertainty in M is a function of the real and imaginary components of Γ_M, their corresponding uncertainties and the correlation coefficient. For example, if we assume that the S_{11} measurements used in the earlier example, and summarized in Table 2, are determinations of Γ_M, then the resulting value for the comparison loss with uncertainty [from (17) and (18)], is $M = 0.9216 \pm 0.0035$. However, for most routine power meter calibrations, $|\Gamma_M|$ is rarely greater than 0.1. (The earlier example was equivalent to $\overline{|S_{11}|} = 0.2800$, which is an unrealistically high value for $|\Gamma_M|$.) Within this constraint (i.e. $|\Gamma_M| \leq 0.1$), the complex value of Γ_M can range over the entire complex plane of values. It is therefore appropriate to examine trends in $u(M)$ as a continuous function of Γ_M within a circle of radius 0.1 centred on the origin in the complex VRC plane (as this corresponds to $0 \leq |\Gamma_M| \leq 0.1$). This can be visualized as a *surface of uncertainty* above the complex VRC plane, with a vertical axis used to depict the size of $u(M)$. The height of this surface will be a function of the complex value of Γ_M, the corresponding uncertainties, $u(x)$ and $u(y)$, and correlation coefficient $r(x, y)$, as described by (18).

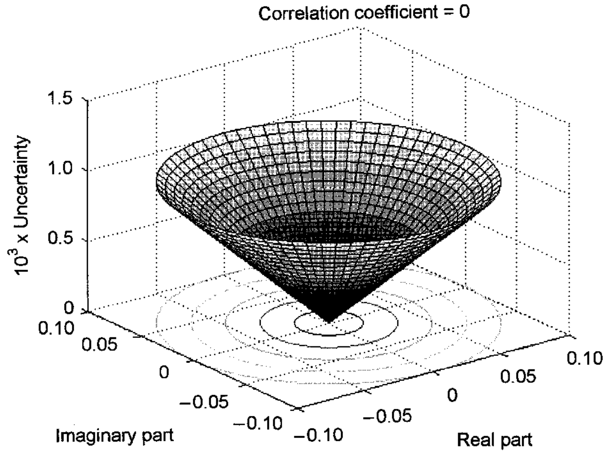

Figure 2. Uncertainty profile for M plotted above the complex Γ_M plane. Uncertainty $u(x) = u(y) = 0.005$, correlation coefficient $r(x, y) = 0$.

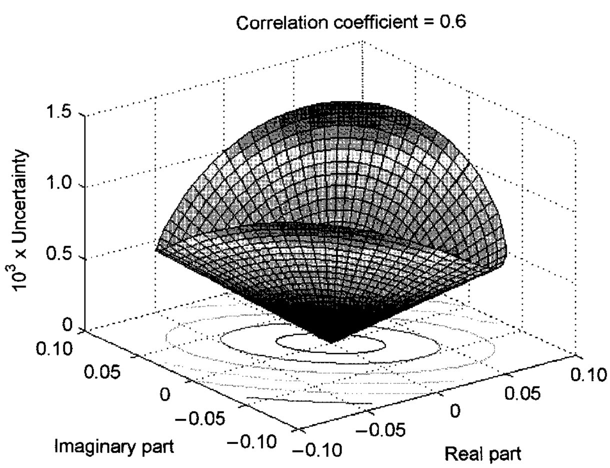

Figure 3. Uncertainty profile for M plotted above the complex Γ_M plane. Uncertainty $u(x) = u(y) = 0.005$, correlation coefficient $r(x, y) = +0.6$.

To illustrate this, two such surfaces are shown in Figures 2 and 3. In both figures it is assumed that $u(x) = u(y) = 0.005$ independent of Γ_M. In Figure 2, $r(x,y) = 0$, and in Figure 3, $r(x,y) = +0.6$. These surfaces show clearly the functional dependency of M on x, y and $r(x,y)$, and further such surfaces could be generated to depict the functional dependency of M on $u(x)$ and $u(y)$. In practice, computer-animation techniques (where the size of a given variable can be varied in real time) provide straightforward and very useful methods for studying trends in uncertainties in such measurements through the use of these surfaces of uncertainty.

Finally, it is worth noting that in both Figures 2 and 3, the uncertainty in M is seen to be zero when $\Gamma_M = 0$. This is an unrealistic feature of (18) that is produced because the law of propagation of uncertainty provides only a linear approximation. Under these circumstances, alternative techniques (such as those described in [10]) are required.

10. Conclusions

This paper has presented methods for evaluating and expressing the uncertainty in complex S-parameter measurements. The methods represent extensions to current international guidelines [1], making them suitable for complex-valued quantities. The methods have been applied to single S-parameters (such as reflection coefficients) as well as multiple S-parameters used to characterize n-port devices. It has been recommended that S-parameters should be expressed, in general, in terms of real and imaginary components, converting to a magnitude and phase format, if required, only after all numerical manipulations have been performed on the data.

A generalized treatment of the uncertainty in complex measurands has included the use of correlation coefficients and covariance matrices to fully describe the uncertainty, along with an indication of the reliability of a sample correlation coefficient. These concepts have been extended to n-port S-parameter measurements. Information has also been given on choosing coverage factors to achieve a given level of confidence for multi-port complex measurements.

Finally, consideration has been given to methods of using complex-valued results to determine other measurement quantities. This has been illustrated by a simple example involving a comparison loss correction for a microwave power meter calibration.

Acknowledgement. The work described in this paper was funded by the National Measurement System Policy Unit of the UK Government Department of Trade and Industry.

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Received on 21 January 2002 and in revised form on 7 February 2002.