ANAMET Report 036
September 2002

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Measuring signals close to the noise floor

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1 Introduction

The presence of noise in a microwave measurement receiver results in an error in the measured magnitude of a CW signal[1]. The error is small if the magnitude of the signal is well above the noise floor of the receiver but gets larger as the signal gets smaller. The size of this error is of interest in a wide range of RF and microwave measurement applications.

In order to investigate the error, a simple mathematical model is introduced in which the CW signal is represented by a constant phasor, the noise is represented by a normally distributed random phasor and the indication of the receiver is given by the sum of the constant phasor and the random phasor. Monte Carlo Simulation can then be used to predict, from the model, how the error in magnitude varies with the signal to noise ratio[2]. The error due to noise predicted by the model is then compared with the error predicted by the coherent and incoherent addition of signal and noise and also with the error observed in practice for a spectrum analyser.

2 A mathematical model of a CW signal in the presence of receiver noise

It will be assumed that the noise voltage in a microwave measurement receiver can be represented by the random phasor

$$N = X + jY$$

whose real and imaginary parts are independently normally distributed with mean 0 and standard deviation σ. This is written

$$X, Y \sim Normal\left(0, \sigma^2\right).$$

N can be thought of as a random vector in the complex plane. Each repeat measurement of the noise is represented as a realisation of the random phasor *N* (see Figure 1).

Under these assumptions:

(i) The magnitude of the noise voltage has a Rayleigh distribution with scale parameter σ. The Rayleigh distribution was first studied by Lord Rayleigh in the 19th century [1]. The distribution has mean 1.25 σ and mode σ. Figure 2 shows a simulation of a Rayleigh distribution with $\sigma = 1$.

(ii) The phase of the noise voltage is uniformly distributed between -180° and +180°. Figure 3 shows a simulation of such a uniform phase distribution.

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[1] If the receiver is a vector receiver, there will be a corresponding error in the measured phase.

[2] The signal to noise ratio is the ratio of the magnitude of the signal to the time averaged magnitude of the receiver noise.

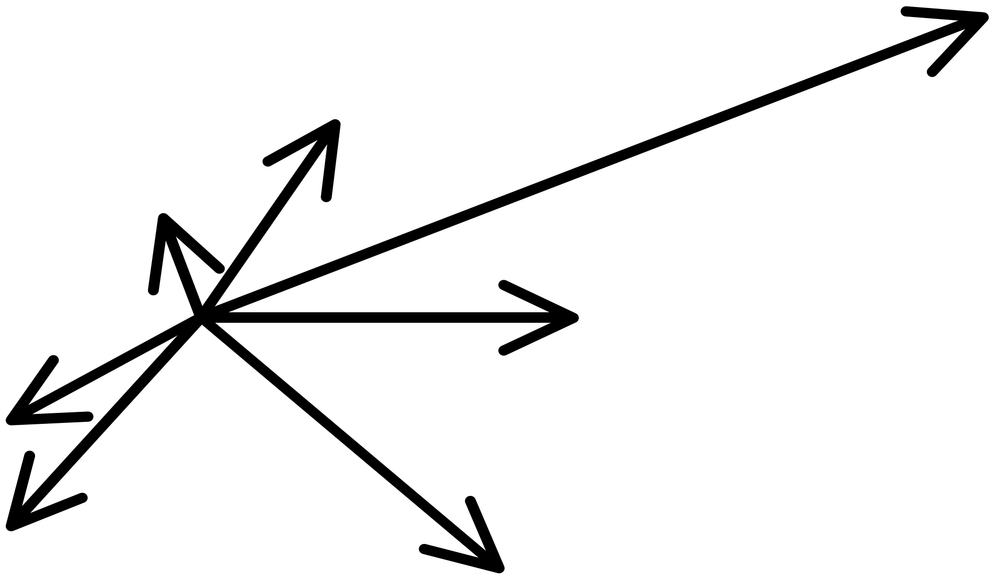

Figure 1: Representation of the random noise phasor $N = X + jY$ as a random vector in the complex plane. Each arrow represents a realisation of the random phasor.

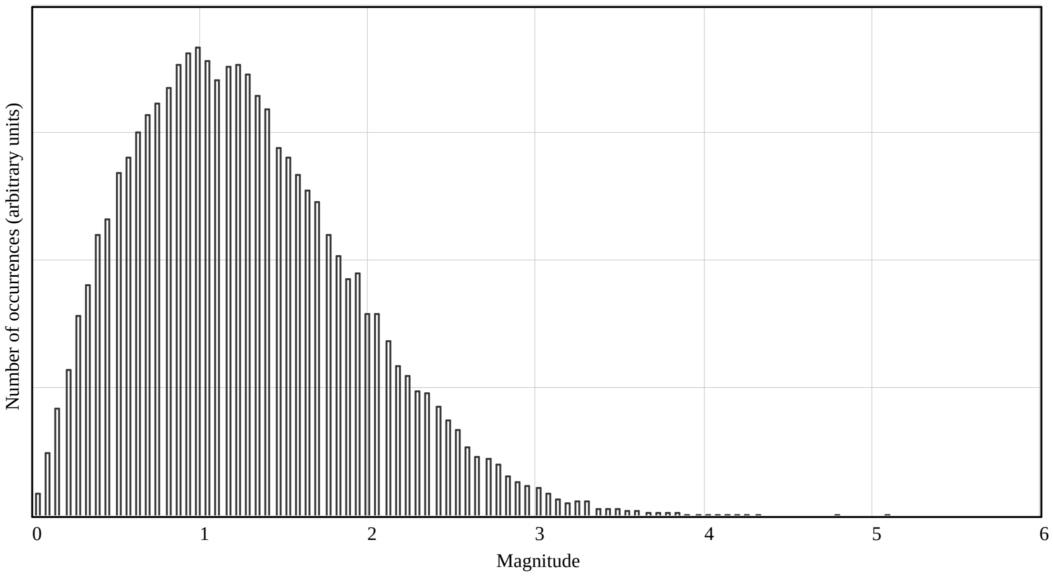

Figure 2: Rayleigh distribution ($\sigma = 1$).

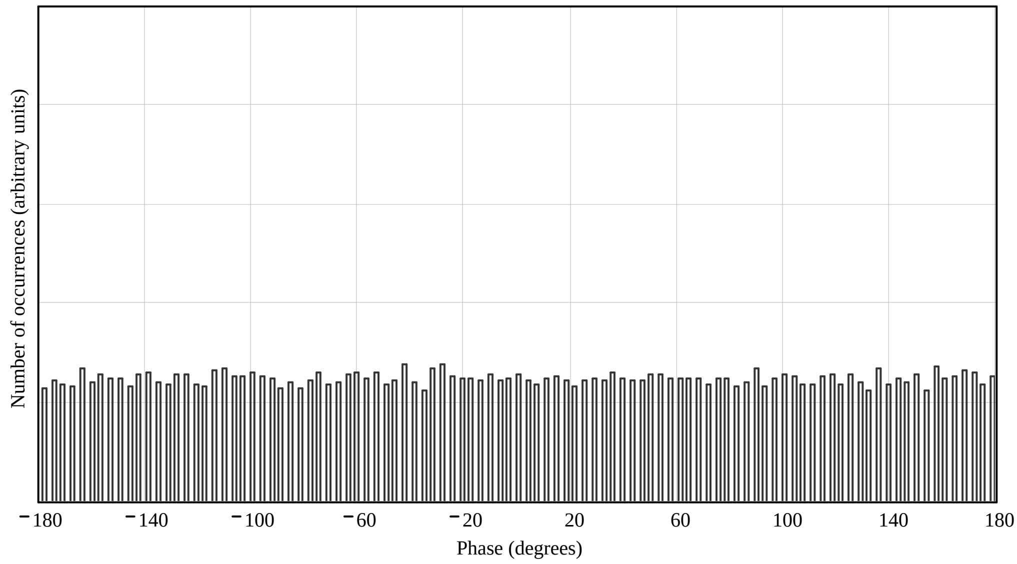

Figure 3: Uniform phase distribution.

Now, consider a CW microwave signal with voltage magnitude *A* and angular frequency ω applied to the input of a measurement receiver (e.g. a spectrum analyser). The instantaneous value of the signal at time *t* is given by the real part of

$$A\exp[j(\omega t + \phi_0)] = A\exp[j\phi_0]\exp[j\omega t]$$

where the initial phase ϕ_0 is determined by an arbitrary choice of the origin of the time axis. The corresponding time-independent phasor

$$s = A\exp(j\phi_0) = A\cos\phi_0 + jA\sin\phi_0$$

can be represented as a two dimensional vector in the complex plane (see Figure 4).

The voltage indicated by the receiver is the resultant of the signal and the noise and so corresponds to the phasor sum

$$R = s + N\ .$$

R is also a random phasor, being the sum of a constant signal phasor (*s*) and a random noise phasor (*N*). It is convenient to choose the origin of the time axis so that $\phi_0 = 0$. If this is done, the constant signal phasor is aligned with the real axis and

$$R = A + (X + jY) = (A + X) + jY = V + jW\ .$$

Each repeat measurement of the signal in the presence of the receiver noise is represented as a realisation of the random phasor *R*. This is shown in Figure 5.

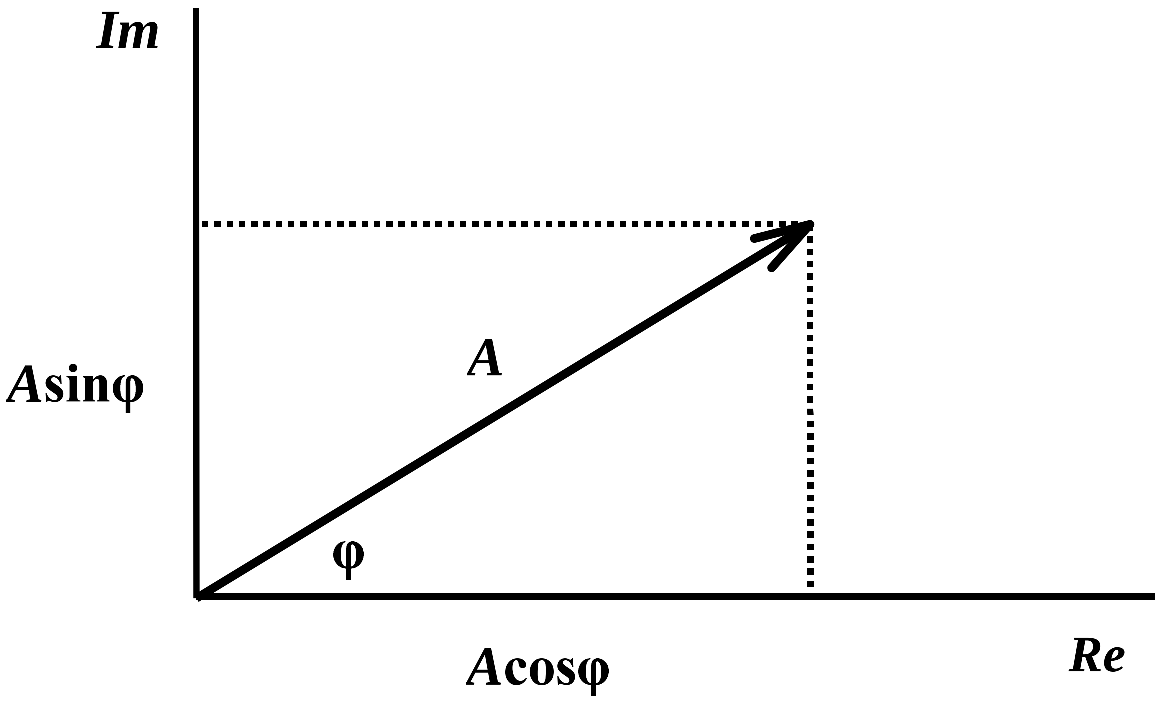

Figure 4: Representation of the signal phasor $s = A\exp(j\phi_0)$ as a two dimensional vector in the complex plane

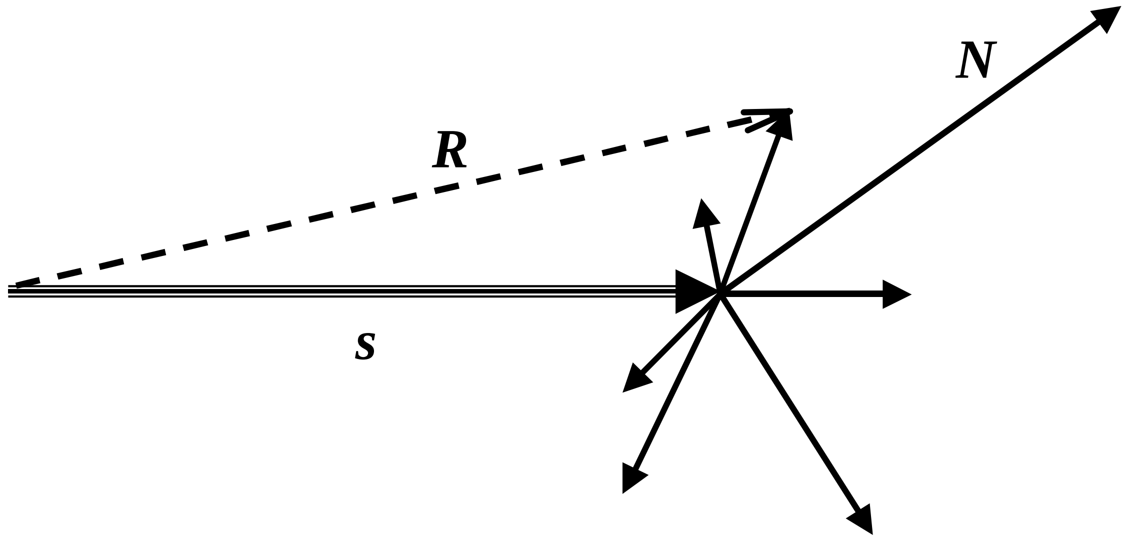

Figure 5: The triple arrow represents the constant signal phasor *s* aligned with the real axis. The solid arrows represent realisations of the random noise phasor *N*. The dashed arrow represents one realisation of the random phasor sum *R*.

The real and imaginary parts of the random phasor sum *R* are independently normally distributed with standard deviation σ. The mean of the real part is *A* and that of the imaginary part is 0. This is written

$$V = A + X \sim \text{Normal}(A, \sigma^2)$$

$$W = Y \sim \text{Normal}(0, \sigma^2)$$

Under these assumptions:

(i) The magnitude of *R* has a Rice distribution with parameters *A* and σ. The Rice distribution was first studied by Rice in the 1940s [2][3]. Figures 6 and 7 show simulations of Rice distributions with $A/\sigma = 1$ and $A/\sigma = 100$ respectively. For small A/σ, the Rice distribution is approximately Rayleigh (it is exactly Rayleigh when $A/\sigma = 0$) whilst, for large A/σ, the Rice distribution is approximately normal[4].

(ii) When A/σ becomes non-zero, the distribution of the phase of *R* is no longer uniform. Figures 8 and 9 show simulations of phase distributions with $A/\sigma = 1$ and $A/\sigma = 100$ respectively. As can be seen from Figure 9, for large A/σ, the phase distribution is narrowly centred on the phase of the signal (0°).

In the model, the ratio of the Rice distribution parameters A/σ is related to the signal to noise ratio. *A* is the magnitude of the signal voltage and, as mentioned above, the mean magnitude of the noise voltage is $1.25\,\sigma$.

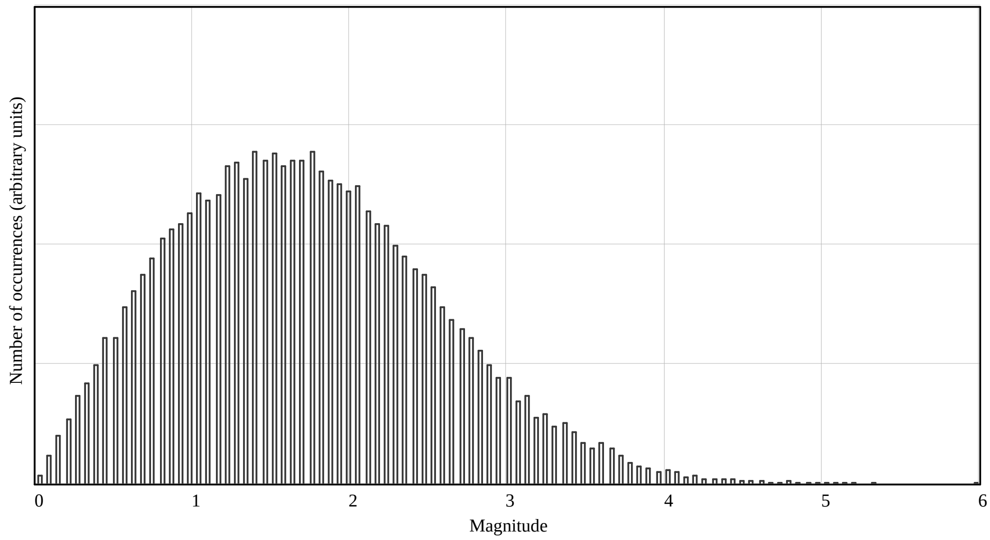

Figure 6: Rice distribution ($A/\sigma = 1$)

[3] Note that the Rice distribution has two parameters (*A* and σ) whereas the Rayleigh distribution has only one parameter (σ).

[4] An equation for the mean (and higher moments) of the Rice distribution is given in [2] (equation (3.10-12)). In [2], the parameters of the distribution are denoted *P* and ψ_0 (corresponding to *A* and σ^2 respectively). The equation is not reproduced here as it is rather complicated.

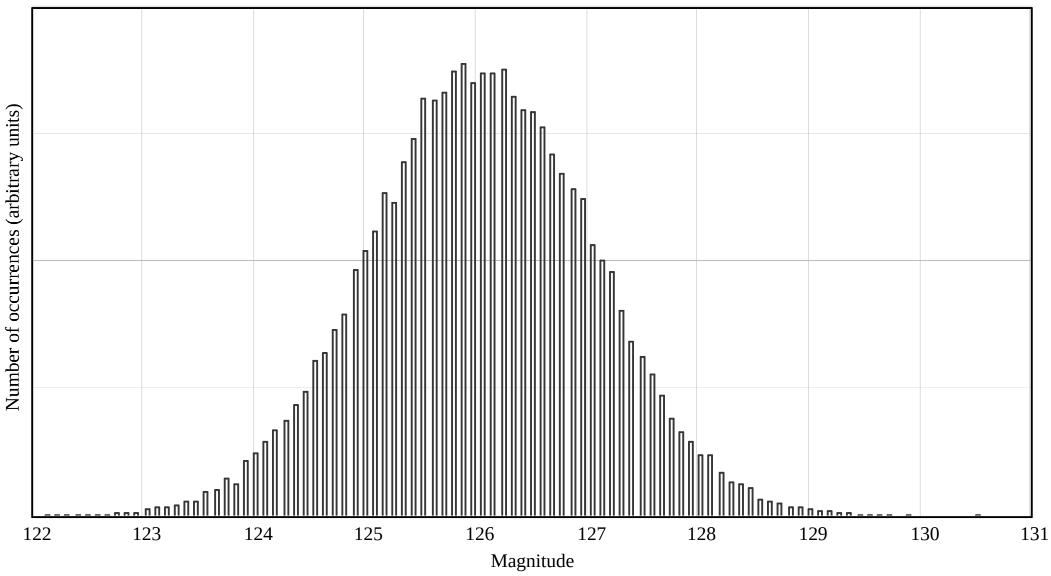

Figure 7: Rice distribution ($A/\sigma = 100$)

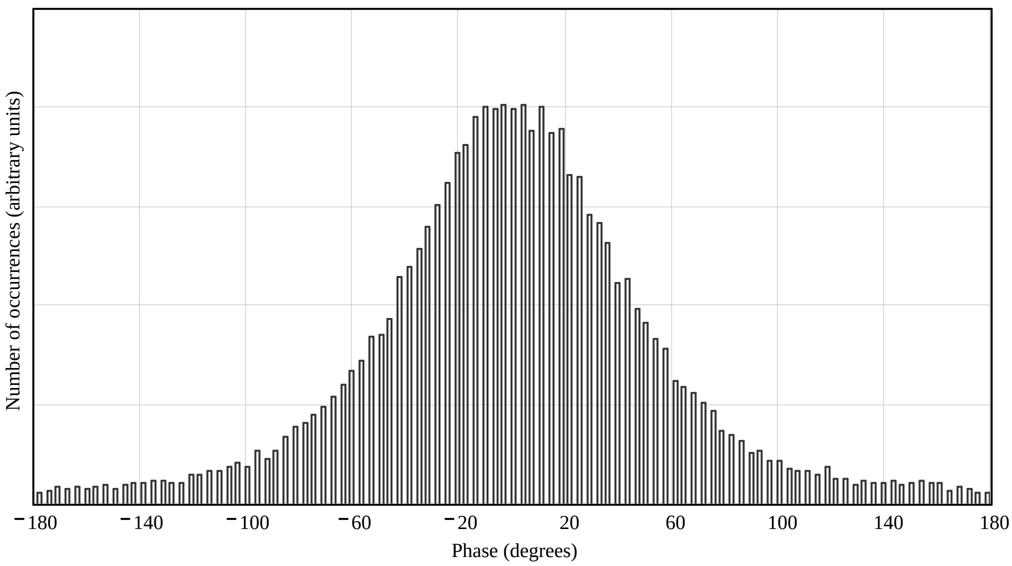

Figure 8: Phase distribution ($A/\sigma = 1$)

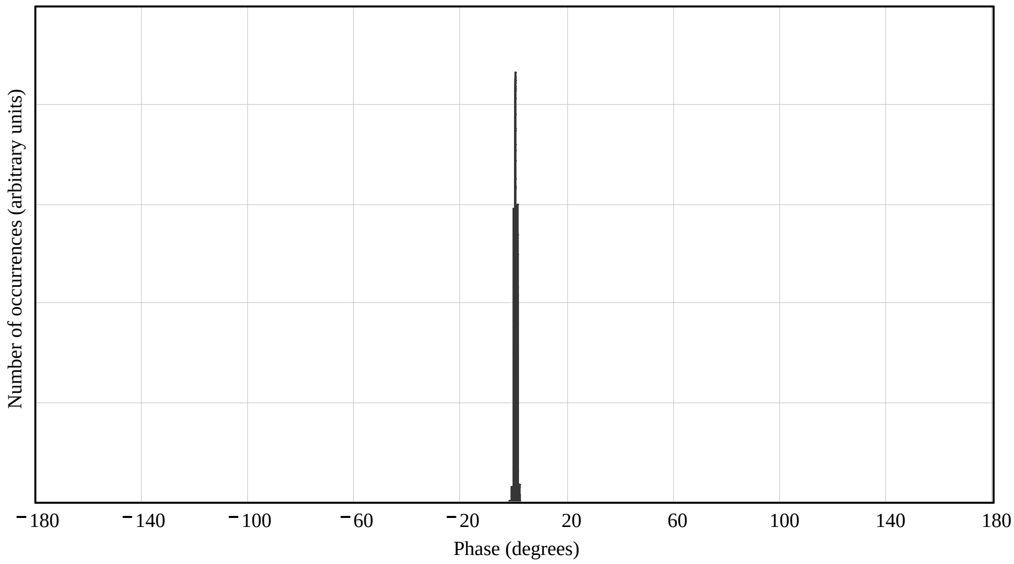

Figure 9: Phase distribution ($A/\sigma = 100$)

3 Analysis of the model using Monte Carlo Simulation

In order to obtain the magnitude error introduced by the receiver noise for a given signal to noise ratio, a Monte Carlo Simulation of the model is performed as follows.

A large sample of *M* vectors[5]

$$\{(X_i, Y_i): i = 1, \ldots, M\}$$

is simulated from a bivariate normal distribution with mean vector (0, 0) and covariance matrix $\begin{pmatrix} \sigma^2 & 0 \\ 0 & \sigma^2 \end{pmatrix}$ [3][6]. This is equivalent to a sample of noise phasors

$$\{N_i = X_i + jY_i : i = 1, \ldots, M\}.$$

The corresponding set of magnitudes $\{|N_i| : i = 1, \ldots, M\}$ is a sample from the Rayleigh distribution with parameter σ. The mean magnitude of the sample of noise phasors $\overline{|N|}$ is taken to be the time-averaged magnitude of the noise voltage in the receiver[7].

[5] A large sample is used to minimise the sample variability. The sample size *M* is typically 50,000.
[6] A convenient choice for σ is σ = 1.
[7] The 'theoretical' result that the mean of the Rayleigh distribution with parameter σ is 1.25 σ is not used directly. However, it does provide a useful check on the simulation.

The constant phasor $A + j\,0$, which represents the CW signal of magnitude A, is added to each noise phasor. In this way a sample of resultant phasors is obtained

$$\{R_i = (A + X_i) + jY_i : i = 1, \ldots, M\}.$$

The corresponding set of magnitudes $\{|R_i| : i = 1, \ldots, M\}$ is a sample from the Rice distribution with parameters A, σ. The mean magnitude of this sample of resultant phasors $\overline{|R|}$ is taken to be the time-averaged voltage magnitude indicated by the receiver[8].

To summarise: the time-averaged magnitude of the noise voltage in the receiver is given by the mean of the Rayleigh distribution with parameter σ and the time-averaged voltage magnitude indicated by the receiver in the presence of a CW signal of magnitude A is given by the mean of the Rice distribution with parameters A, σ. These mean values are obtained by simulation as described above.

In terms of quantities set or determined during the Monte Carlo Simulation, the signal to noise ratio is $\dfrac{A}{\overline{|N|}}$ and the error expressed in decibels is $20\log_{10}\dfrac{\overline{|R|}}{A}$.

The predicted error in the measured magnitude of a CW signal due to noise for several different values of the signal to noise ratio is shown in Table 1. The values were obtained by Monte Carlo Simulation of the mathematical model as described above. According to these predictions, a signal with voltage magnitude one half of the time averaged noise voltage magnitude (signal to noise ratio –6 dB) gives rise to an error of 6.8 dB whereas a signal equal in magnitude to the noise (signal to noise ratio 0 dB) gives rise to an error of 2.7 dB. When the magnitude of the signal voltage is one hundred times that of the noise voltage (signal to noise ratio 40 dB) the error is only 0.00035 dB.

Table 1: Predicted error in the measured magnitude due to presence of noise in the receiver for different values of the signal to noise ratio

Signal to noise ratio (dB)	Error in measured signal magnitude (dB)
-6	6.8
0	2.7
3	1.4
6	0.71
10	0.28
15	0.089
20	0.028
40	0.00035

[8] The ‘theoretical’ formula, referred to earlier, for the mean of the Rice distribution could be used here as a check on the simulation.

4 Comparison of the model with coherent and incoherent addition of signal and noise

In addition to the use of the mathematical model described above together with Monte Carlo Simulation, two other methods of combining the signal and noise are now considered. These are referred to as 'coherent addition' and 'incoherent addition'.

For coherent addition, the noise is treated as a CW signal of unknown phase. The combination of signal and noise is obtained by phasor addition. Since the phase difference is unknown, the two extreme cases of in-phase and anti-phase are considered. If the magnitude of the signal voltage is $|S|$ and the time-averaged magnitude of the noise voltage is $|N|$, the resultant voltage magnitude $|R|$ as indicated by the receiver is given, for the in-phase case, by

$$|R| = |S| + |N|$$
$$\frac{|R|}{|S|} = \frac{|S|/|N| + 1}{|S|/|N|}$$

and, for the anti-phase case, by

$$|R| = \big||S| - |N|\big|$$
$$\frac{|R|}{|S|} = \frac{\big||S|/|N| - 1\big|}{|S|/|N|}.$$

For incoherent addition, the resultant indicated voltage magnitude $|R|$ is obtained by adding the magnitudes of the signal and of the noise in quadrature

$$|R| = \sqrt{|S|^2 + |N|^2}$$
$$\frac{|R|}{|S|} = \frac{\sqrt{(|S|/|N|)^2 + 1}}{(|S|/|N|)}$$

In each of the above three cases, $|S|/|N|$ is the signal to noise ratio and the error due to the noise expressed in decibels is given by

$$Error\ (dB) = 20\log_{10}\left(\frac{|R|}{|S|}\right)$$

Figure 10 shows the error due to the noise as a function of the signal to noise ratio as calculated by (i) Monte Carlo simulation, (ii) in-phase coherent addition, (iii) anti-phase coherent addition and (iv) incoherent (quadrature) addition. From the figure, it can be seen that the curves obtained by simulation and by incoherent addition show good agreement. On the other hand, the curves obtained by coherent addition are quite different.

5 Comparison of the model with measurement

In order to measure the error due to noise as a function of the signal to noise ratio for a spectrum analyser, the indicated magnitude is observed for several applied signals of known magnitude. The known signal levels are achieved using a signal generator and a calibrated switched attenuator. The amplitude of the signal generator is set in order that, with the calibrated attenuator switched to zero attenuation, the signal level is sufficiently far above the noise floor of the spectrum analyser to be accurately measured. Increasing the attenuation of the attenuator in known steps allows signals of several known amplitudes to be applied to the spectrum analyser. The error is obtained from the known true magnitude and the magnitude indicated by the spectrum analyser. The level of the noise floor is measured with no applied signal and is used to calculate the signal to noise ratio. All the spectrum analyser readings are obtained as the average of a number of repeat measurements. A schematic diagram of the equipment used is shown in Figure 11.

Figures 12 and 13 show the measured error plotted against the signal to noise ratio. In Figure 12, a CW signal was used. This is the situation which has been discussed hitherto. In Figure 13, a frequency comb was used and attention was restricted to one of the ‘spikes’. In both Figures, good agreement is observed between the measured error curve and the error curve predicted by Monte Carlo Simulation. This shows that the effect of noise on each component of a frequency comb can be treated as though the component were a CW signal.

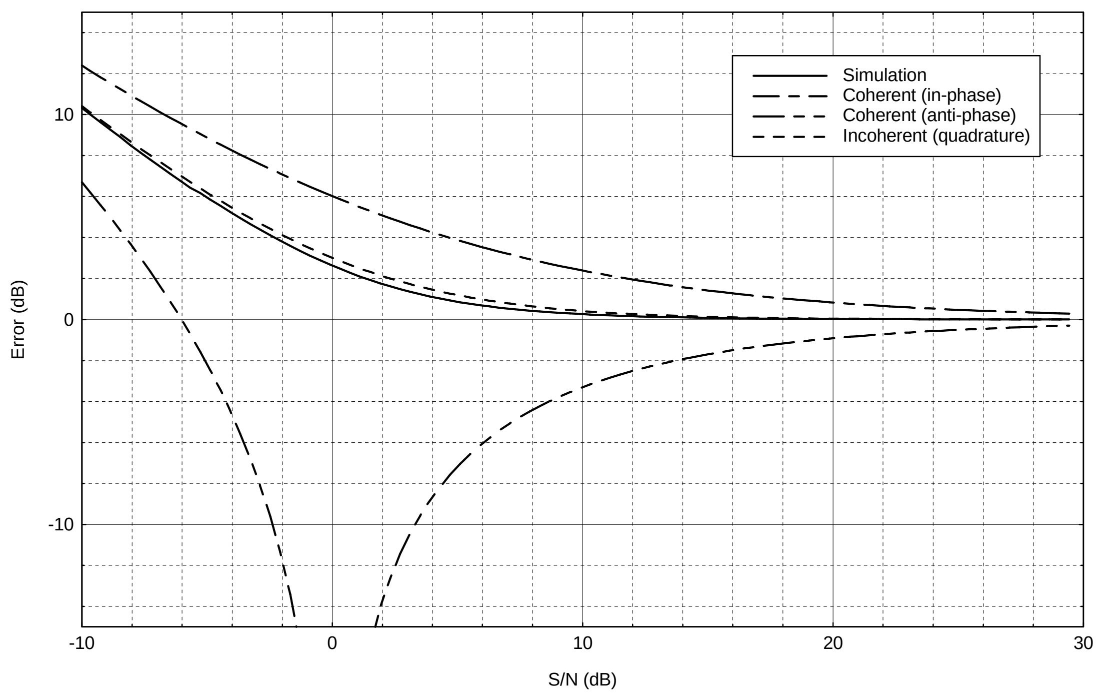

Figure 10: Predicted error in magnitude due to noise as a function of the signal to noise ratio

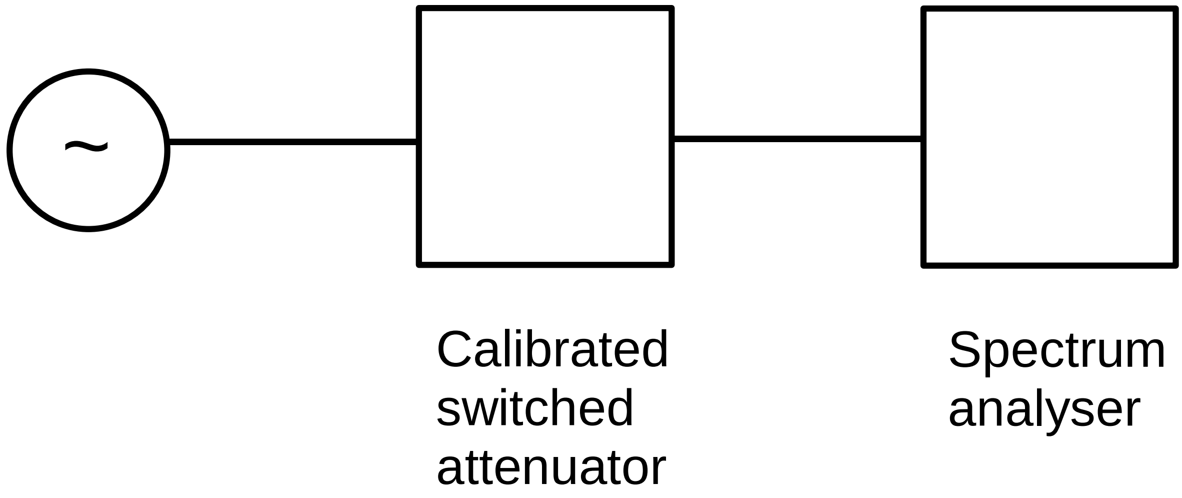

Figure 11: Schematic diagram of system used to measure error due to noise for a spectrum analyser as a function of signal to noise ratio

6 Conclusion

A straightforward and intuitive mathematical model of a CW microwave signal in the presence of noise has been presented. Monte Carlo Simulation has been used in conjunction with the model to predict the variation of the error in indicated magnitude with the signal to noise ratio in a receiver. The error curve thus obtained shows good agreement with that obtained by the quadrature addition of signal and noise and also with the observed behaviour of a spectrum analyser. According to the model, the error decreases from about 3 dB when the signal to noise ratio is 0 dB to about 0.3 dB when the signal to noise ratio is 10 dB. For a signal to noise ratio greater than about 15 dB, the error is less than 0.1 dB.

References

[1] J W S Rayleigh, "On the resultant of a large number of vibrations of the same pitch and arbitrary phase", Philosophical Magazine, 5th Series, Vol. 10, pp 73-78, 1880.

[2] S O Rice, "Mathematical Analysis of Random Noise", Bell System Technical Journal, Vol. 23, pp 282-333, July 1944; Vol. 24, pp 96-157, January 1945.

[3] M J Salter, N M Ridler and M G Cox, "Distribution of correlation coefficient for samples taken from a bivariate normal distribution", *NPL Report CETM 22*, September 2000.

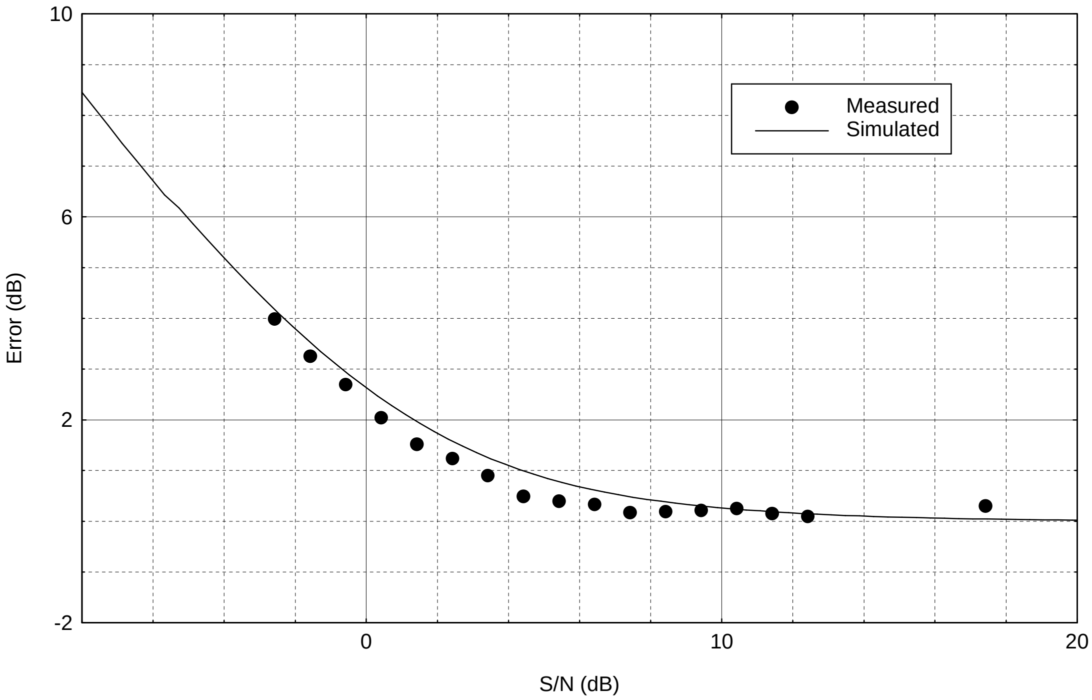

Figure 12: Measured and simulated error due to noise for a spectrum analyser (measurement used a CW source)

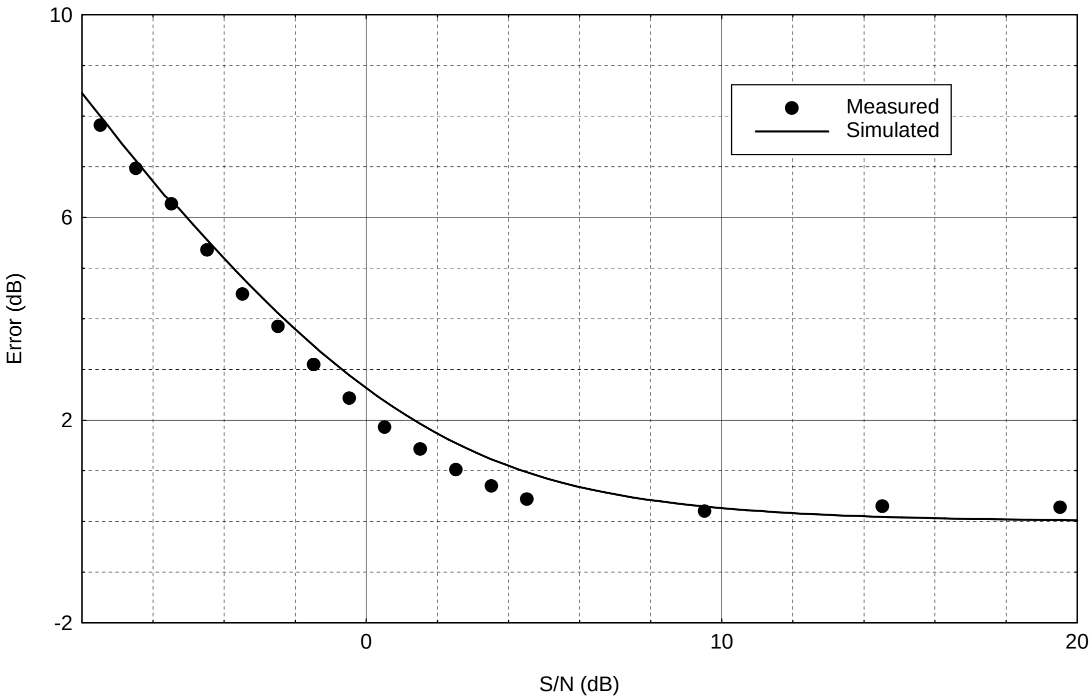

Figure 13: Measured and simulated error due to noise for a spectrum analyser (measurement used a comb generator)