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Converting between logarithmic and linear formats for reflection and transmission coefficients.
Part 3: propagation of distributions

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**Converting between logarithmic and linear formats for reflection and transmission coefficients. Part 3: propagation of distributions**

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## Introduction

This report continues the discussion, begun in ANA*Tips* 4 and 5 [1, 2], of logarithmic and linear formats for expressing the magnitude of an *S*-parameter and the corresponding uncertainties. Recall that for a complex-valued *S*-parameter $S_{ij}$, the linear magnitude is $M_{\mathrm{LIN}} = |S_{ij}|$ and the logarithmic magnitude is $M_{\mathrm{LOG}} = -20\log_{10}|S_{ij}|$.

Two approaches to the problem of converting *S*-parameter uncertainty intervals from linear magnitude format to logarithmic magnitude format, and vice versa, were considered in [1, 2]:

(i) use of the law of propagation of uncertainty
(ii) direct conversion of the endpoints of the uncertainty interval.

A third approach is considered here which is based on Monte Carlo simulation and which could be called the “propagation of distributions”. Starting from the assumption that the real and imaginary parts of the *S*-parameter are jointly normally distributed, the corresponding distributions of linear magnitude and logarithmic magnitude are simulated and conclusions about the uncertainties are drawn from the distributions.

A data simulator [3] is used to generate a large random sample of 2-dimensional vectors (complex numbers) from a population with a bivariate normal distribution [4]. These vectors are then used to model repeat measurements of an *S*-parameter. A large sample is chosen so that the distribution of a sample property (such as the linear magnitude or the logarithmic magnitude) is a good approximation to the distribution of the corresponding population property. The population bivariate normal distribution is specified by its 2-dimensional mean vector and its 2 × 2 covariance matrix. The diagonal elements of the covariance matrix are the variance (i.e. squared standard deviation) of the real part $\left(\sigma_x{}^2\right)$ and the variance of the imaginary part $\left(\sigma_y{}^2\right)$. The matrix is symmetric and the two off-diagonal elements are equal to the covariance between the real and imaginary parts $\left(\sigma_{xy} = \sigma_{yx}\right)$. The correlation coefficient between real and imaginary parts is defined as follows:

$$\rho_{xy} = \frac{\sigma_{xy}}{\sigma_x \sigma_y}$$

The distribution can be represented geometrically by an ellipse. If $X$ is a variable vector in the plane, $\mu$ is the population mean vector and $V$ is the population covariance matrix then the following equation defines an ellipse in the plane which encompasses 39% of the population:

$$(X - \mu)^{\mathrm{T}} V^{-1} (X - \mu) = 1$$

In this equation superscript T denotes vector transpose and superscript –1 denotes matrix inverse. If the two variances are equal and the covariance is zero (in which case the covariance matrix is a multiple of the identity matrix) then the ellipse degenerates into a circle centred on the mean value and with radius equal to the square root of the variance (i.e. the standard deviation).

In the simulation, the population standard deviations of the real and imaginary parts correspond to the standard uncertainties in the measurement of the real and imaginary parts of the *S*-parameter. In practice the measured values might be obtained as the mean of a number of repeat measurements in which case the population distribution is the distribution of mean values and the population standard deviations are "standard errors in the mean".

Starting from the assumption of a normal distribution, a simulation allows the distributions of linear magnitude and logarithmic magnitude to be determined. To do this experimentally would require an impractically large number of repeat measurements to be taken.

**Example 1 (Example 1 in ANA*Tips* 4 and ANA*Tips* 5)**

Example 1 in [1, 2] can be stated as follows:
*"The measured* $|S_{11}|$ *of a nominal 2.0 VSWR mismatch termination was found to be 0.3288 ± 0.0078. Find the equivalent return loss and uncertainty in return loss."*

For this example, where $|S_{11}|$ is large relative to its uncertainty, there is good agreement between the uncertainty intervals obtained for the logarithmic magnitude by methods (i) and (ii) mentioned above as is shown in [2]. The equivalent return loss (logarithmic magnitude) and uncertainty are found to be (9.66 ± 0.21) dB.

To investigate this example further a large random sample (of size 100,000) was simulated from a bivariate normal distribution. The population mean was chosen as 0.3288 + j 0 which has magnitude equal to $|S_{11}|$. Note that any complex number with magnitude equal to $|S_{11}|$ could have been used as the population mean; a value on the real axis was chosen for simplicity. The population standard deviations in the real and imaginary parts were both set to σ = 0.0078 which is equal to the uncertainty in $|S_{11}|$. This is consistent with the law of propagation of uncertainty if the uncertainty in $|S_{11}|$ is taken as a standard uncertainty. The population correlation coefficient between the real and imaginary parts was set to zero for simplicity. The population covariance matrix corresponding to these population parameters is a multiple of the identity matrix (the variances are equal to $\sigma^2$ and the covariances are zero) and corresponds to a circular region in the complex plane centred at the population mean value and with radius equal to σ.

The distributions of linear magnitude and logarithmic magnitude in the sample are presented in Figs. 1 and 2. The mean, standard deviation and an interval which encompasses 95% of the distribution are given in Table 1 for both distributions. In each case the 95% interval is obtained by trimming 2.5% of the distribution from each end.

Table 1 Parameters of linear and logarithmic magnitude distributions for Example 1

| | Linear magnitude | Logarithmic magnitude |
|---|---|---|
| Mean | 0.3289 | 9.66 |
| Standard deviation | 0.0078 | 0.21 |
| 95% Interval | (0.3136, 0.3441) | (9.26, 10.07) |

As can be seen from Figs. 1 and 2 the distributions of linear magnitude and logarithmic magnitude are both symmetric. The means and standard deviations of the two distributions agree with the values and uncertainties [given for linear magnitude and calculated by methods (i) and (ii) for logarithmic magnitude]. Also for both distributions it was found that about 68.5% of the distribution is within one standard deviation of the mean. In addition, for both distributions the 95% interval is symmetric about the mean.

**Example 2 (Example 2 in ANA*Tips* 5)**

Example 2 in [2] can be stated as follows:

*"The measured $|S_{11}|$ of a nominal near-matched load was found to be 0.0062 ± 0.0054. Find the equivalent return loss and uncertainty in return loss."*

For this example, where $|S_{11}|$ is comparable in size to its uncertainty, methods (i) and (ii) mentioned above give different uncertainty intervals for logarithmic magnitude as shown in [2]. Method (i) gives a symmetric uncertainty interval whereas method (ii) gives an asymmetric interval. The equivalent return loss (logarithmic magnitude) and uncertainty are found to be (44 ± 8) dB [method (i)] and ( $44\left\{\pm\begin{matrix}18\\5\end{matrix}\right\}$ ) dB [method (ii)].

This example was further investigated in the same way as Example 1 by simulating a large random sample from a bivariate normal distribution. The population mean was chosen as 0.0062 + j 0 (the point on the positive real axis with magnitude equal to the given $|S_{11}|$) and the population standard deviations in the real and imaginary parts were set equal to 0.0054 (the given uncertainty in $|S_{11}|$ which is therefore being treated as a standard uncertainty). As before the population correlation coefficient between the real and imaginary parts was set to zero.

The distributions of linear magnitude and logarithmic magnitude in the sample are presented in Figs. 3 and 4. The mean, standard deviation and an interval which encompasses 95% of the distribution are given in Table 2 for both distributions. In each case the 95% interval is obtained by symmetric trimming as in Example 1.

Table 2 Parameters of linear and logarithmic magnitude distributions for Example 2

| | Linear magnitude | Logarithmic magnitude |
|---|---|---|
| Mean | 0.0088 | 42.40 |
| Standard deviation | 0.0043 | 5.35 |
| 95% Interval | (0.0017, 0.0182) | (34.81, 55.54) |

As can be seen from Figs. 3 and 4 the distributions of linear magnitude and logarithmic magnitude are both skew. In fact, the logarithmic magnitude distribution is more skew than the linear magnitude distribution (as measured by their skewness parameter which is related to the so-called third central moment of the distribution). For both distributions the mean does not occur at the peak of the distribution as it would for a normal distribution.

The means of the distributions (0.0088 for linear magnitude and 42.40 dB for logarithmic magnitude) differ from the values of linear magnitude and logarithmic magnitude which are calculated from the mean real and imaginary parts (0.0062 and 44 dB). The standard deviations of the distributions (0.0043 and 5.35 dB) differ from the uncertainties in linear magnitude and logarithmic magnitude given above (± 0.0054 for linear magnitude and ± 8 dB or $\left\{\pm\begin{matrix}18\\5\end{matrix}\right\}$ dB for logarithmic magnitude). The uncertainty in linear magnitude is calculated from the uncertainties in real and imaginary parts by propagation of uncertainties whilst the uncertainty in logarithmic magnitude is calculated from the uncertainty in linear magnitude by either of methods (i) or (ii) mentioned above.

For the linear magnitude distribution it was found that about 66.7% of the distribution is within one standard deviation of the mean whilst for the logarithmic magnitude the corresponding figure is 73.2%. In addition, for both distributions the 95% interval is asymmetric about the mean.

**Example 3 An example with some surprising properties**

In Example 2 the distributions of linear magnitude and logarithmic magnitude were investigated in the case when the *S*-parameter is close to the origin in the complex plane. The limiting case when the *S*-parameter is actually at the origin is considered in this third example.

*"The measured $|S_{11}|$ of a nominal near-matched load was found to be 0.0000 ± 0.0054. Find the equivalent return loss and uncertainty in return loss."*

The equivalent return loss (logarithmic magnitude) is found to be $+\infty$ dB.

To investigate this example further a large random sample was simulated from a bivariate normal distribution with the following parameters:

population mean: 0 + j 0
population standard deviation in the real part: 0.0054
population standard deviation in the imaginary part: 0.0054
population correlation coefficient between real and imaginary parts: 0

As can be seen from Figs. 5 and 6, the distributions of linear magnitude[*] and logarithmic magnitude in this third example are similar to those in Example 2. The amount by which the given linear magnitude (0.0000) or calculated logarithmic magnitude (+∞ dB) differ from the means of their respective distributions (0.0067 and 44.87 dB) is even greater than in Example 2. In fact, rather surprisingly, it can be seen that the “true values”[#] have been banished to the extreme tails of the distributions.

## Discussion

In the above examples the distributions of real and imaginary parts are symmetric (they are assumed to be normal) and the mean is a natural choice for the representative value. On the other hand, in general, the distributions of linear magnitude and logarithmic magnitude can be skew because these quantities can only assume a restricted range of values. For a passive device:

$$0 \le M_{\mathrm{LIN}} \le 1$$

$$0 \le M_{\mathrm{LOG}} \le \infty$$

The skewness of the distributions increases as the *S*-parameter approaches the origin of the complex plane.

The means of the logarithmic magnitude and the linear magnitude differ from the “true values” obtained by transforming the means of the real and imaginary parts. This effect becomes more pronounced as the *S*-parameter approaches the origin. This suggests the need to always measure real and imaginary parts and to transform to linear magnitude or logarithmic magnitude as required. If this is done then since the distributions of both linear magnitude and logarithmic magnitude are skew, in general, the confidence intervals for both quantities will be asymmetric.

## Recommended method of obtaining uncertainties in linear magnitude and logarithmic magnitude formats

The above examples and discussion suggest the following method of obtaining uncertainties in linear magnitude and logarithmic magnitude formats for an *S*-parameter at a specified confidence level:

(1) Measure the real and imaginary parts of the *S*-parameter and obtain the corresponding standard uncertainties and correlation coefficient.

(2) Calculate the corresponding linear magnitude and logarithmic magnitude ($M_{\mathrm{LIN}}$ and $M_{\mathrm{LOG}}$).

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[*] In this example the linear magnitude has a so-called “Rayleigh distribution”.

[#] The “true values” of the real and imaginary parts are taken to be the means of the corresponding symmetric distributions. The “true values” of linear magnitude and logarithmic magnitude are taken to be the values calculated from the true values of the real and imaginary parts. In general, these differ from the means of the corresponding distributions.

(3) Simulate a large random sample of 2-dimensional vectors (complex numbers) from a population with a bivariate normal distribution. In the simulator, put the complex-valued population mean equal to the measured *S*-parameter, make the population standard deviations in the two components equal to the standard uncertainties in the real and imaginary parts of the *S*-parameter (possibly different), and set the population correlation coefficient between the two components equal to the measured correlation coefficient[†].

(4) Obtain the distribution of linear magnitude and logarithmic magnitude in the sample.

(5) To calculate an α% confidence interval for linear magnitude or logarithmic magnitude proceed as follows. Obtain an interval by trimming (100 - α/2)% from each end of the appropriate distribution (two-sided trimming). If this interval includes the value ($M_{LIN}$ or $M_{LOG}$) take it as the required confidence interval. If the above interval does not include the value then obtain a second interval which does include the value by trimming (100 - α)% from one end of the distribution (one-sided trimming) and take this as the required confidence interval.

The 95% confidence intervals for linear magnitude and logarithmic magnitude derived using this method for Examples 1 to 3 are summarised in Table 3. Only for Example 3 was it necessary to use one-sided trimming. For Example 2 asymmetric intervals are obtained for both linear magnitude and logarithmic magnitude. The results for Example 3 are interpreted as meaning that one has 95% confidence that the nominal near-matched load has reflection coefficient magnitude less than 0.0132 and similarly that one has 95% confidence that it has return loss greater than 38 dB.

Table 3 95% confidence intervals for linear magnitude and logarithmic magnitude

| | Linear magnitude | Logarithmic magnitude |
|---|---|---|
| Example 1 | $0.3288 \pm 0.0153$ | $(9.66 \pm 0.41)$ dB |
| Example 2 | $0.0062 \left\{ \pm \begin{matrix} 0.0120 \\ 0.0045 \end{matrix} \right\}$ | $\left( 44 \left\{ \pm \begin{matrix} 12 \\ 9 \end{matrix} \right\} \right)$ dB |
| Example 3 | $0.0000 \left\{ \pm \begin{matrix} 0.0132 \\ 0.0000 \end{matrix} \right\}$ | > 38 dB |

## Conclusion

A method has been proposed for calculating confidence intervals for the linear magnitude and the logarithmic magnitude of a complex-valued *S*-parameter from the standard uncertainties in the real and imaginary parts. The method is based on Monte Carlo Simulation. In general, the resulting intervals are not symmetric about the value. The method could be extended to obtain confidence intervals for other quantities such as *S*-parameter phase and VSWR.

[†] Alternatively the population correlation coefficient could be set to zero for simplicity as was done in the discussion of Examples 1 to 3.

## References


[1] Nick M. Ridler, “Converting between logarithmic and linear formats for reflection and transmission coefficients”, ANA*Tips* No. 4, October 2000

[2] Nick M. Ridler, “Converting between logarithmic and linear formats for reflection and transmission coefficients. Part 2: when the uncertainty of measurement is relatively large”, ANA*Tips* No. 5, November 2000

[3] M J Salter, N M Ridler and M G Cox, “Distribution of correlation coefficient for samples taken from a bivariate normal distribution”, *NPL Report CETM 22*, September 2000.

[4] K V Mardia, J T Kent and J M Bibby, “Multivariate analysis”, *Academic Press*, 1979.

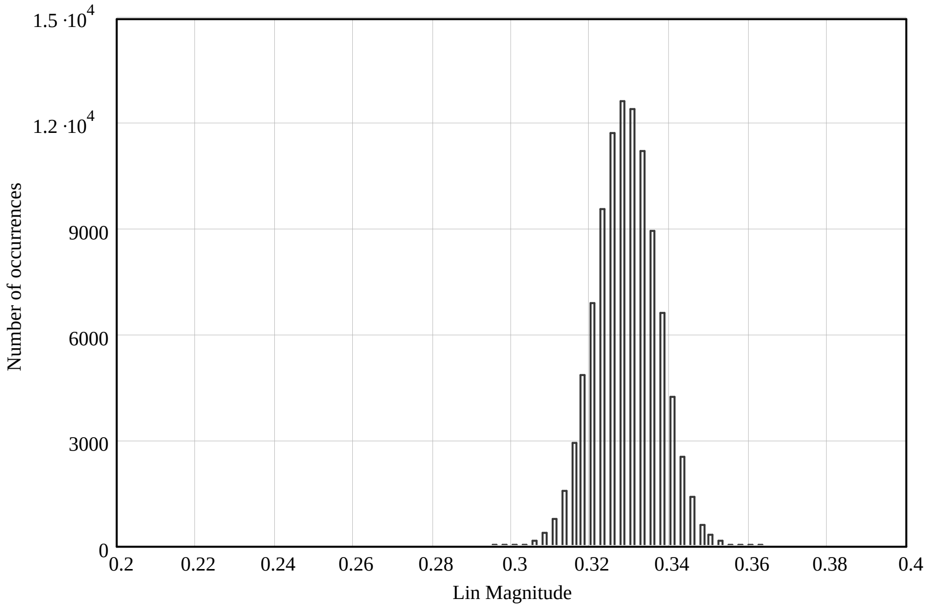


Fig. 1. Distribution of linear magnitude for the random sample of 100,000 points in Example 1

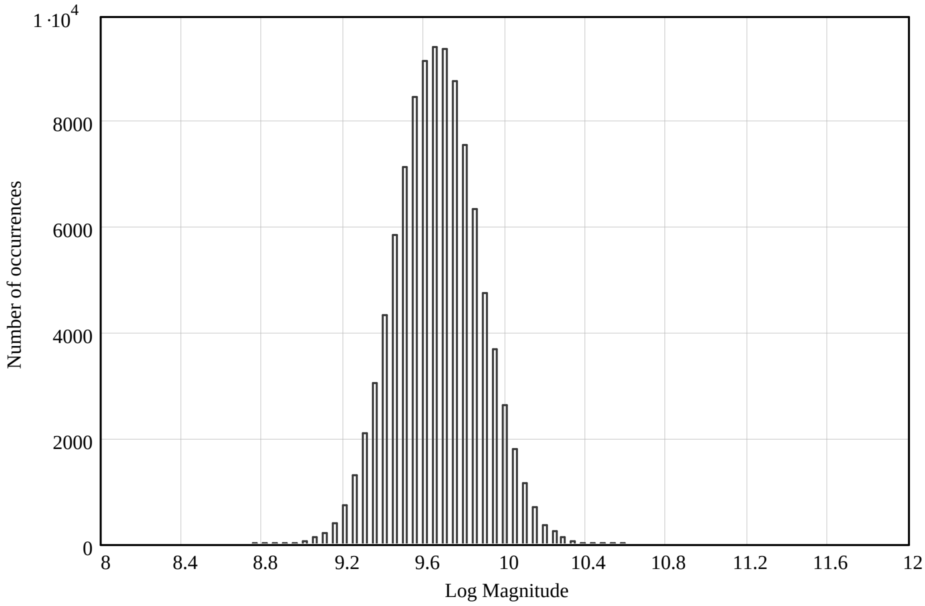


Fig. 2. Distribution of logarithmic magnitude for the random sample of 100,000 points in Example 1

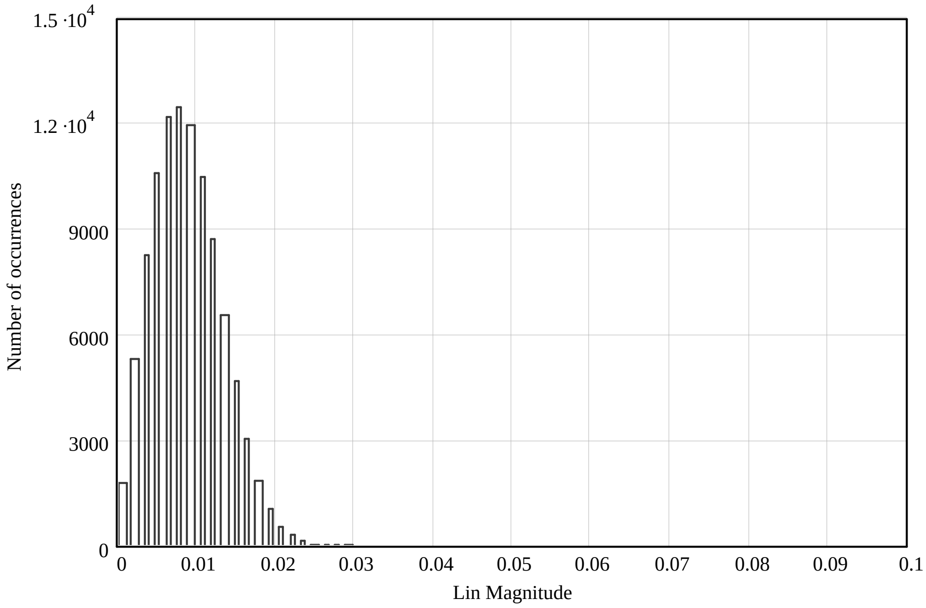


Fig. 3.  Distribution of linear magnitude for the random sample of 100,000 points in Example 2

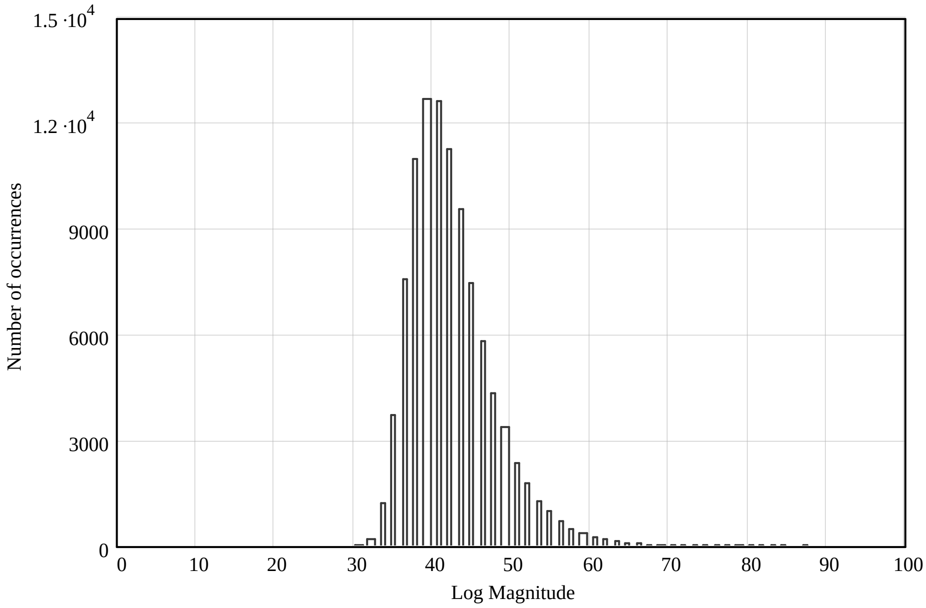


Fig. 4.  Distribution of logarithmic magnitude for the random sample of 100,000 points in Example 2

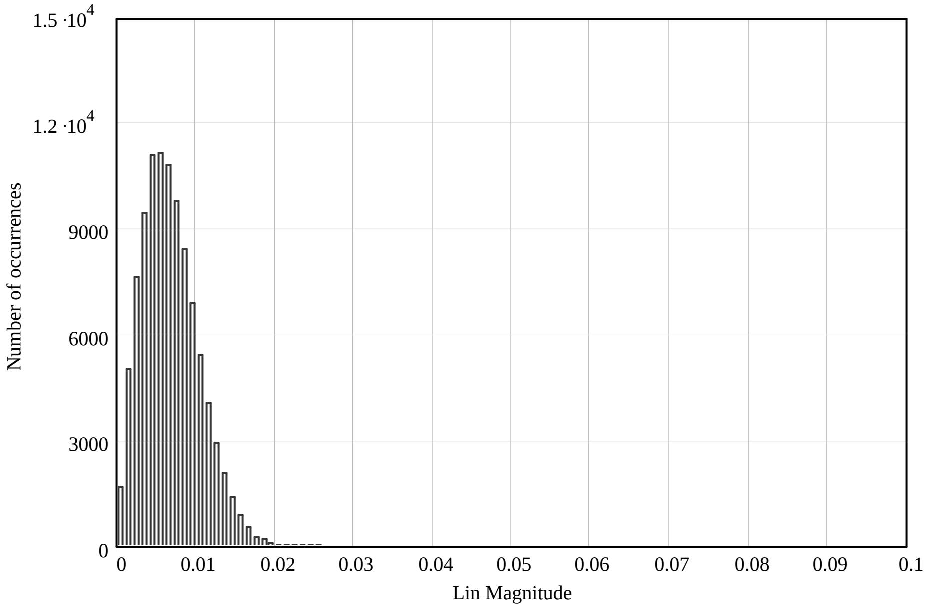


Fig. 5. Distribution of linear magnitude for the random sample of 100,000 points in Example 3

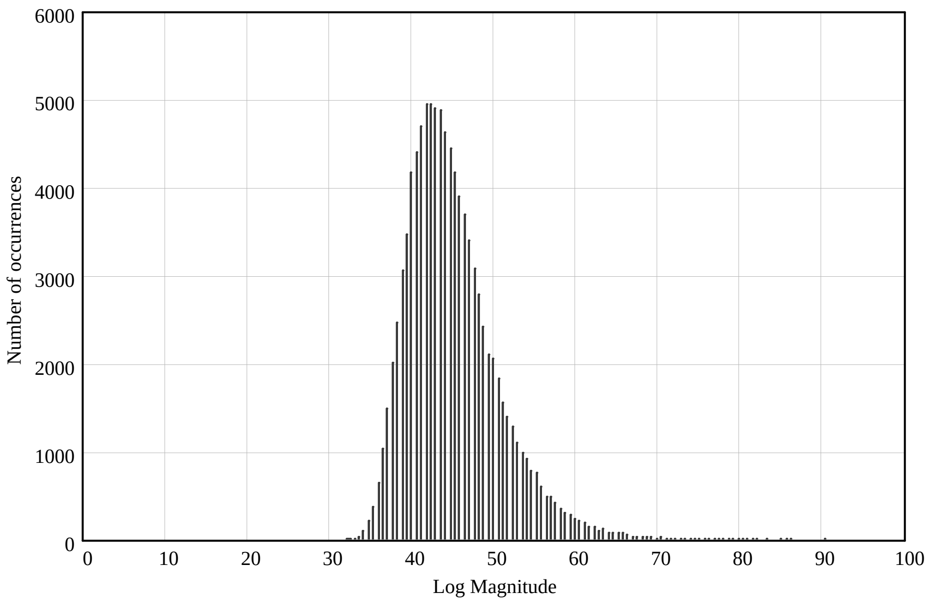


Fig. 6. Distribution of logarithmic magnitude for the random sample of 100,000 points in Example 3