Coverage factors for complex-valued measurands

Martin Salter
National Physical Laboratory

Introduction

In RF and microwave metrology measurands such as complex reflection and transmission coefficients are 2-dimensional (2-D). The result of the measurement of such a 2-D quantity can be expressed as a 2-D mean vector $\overline{X}$ (giving the value) and a 2 X 2 covariance matrix in the mean V (giving the uncertainty). $\overline{X}$ and V define an elliptical uncertainty region in the plane centred on the mean value. This "tip" explains how, using statistical tables, to obtain the coverage factor k such that the matrix k^2V defines an elliptical uncertainty region at a specified level of confidence. Apart from the fact that the elements of V have the dimensions of (uncertainty)^2, V is analogous to standard uncertainty, k^2V is analogous to expanded uncertainty and the elliptical confidence regions are analogous to confidence intervals in one dimension. It is assumed that the underlying distribution is a bivariate normal distribution. It is also assumed that the covariance matrix of the underlying distribution is known to a good approximation i.e. that the effective number of degrees of freedom associated with the measurement is large.

Calculation of coverage factors

Suppose that X is a normally distributed 2-D random vector with population mean vector ì and population covariance matrix Ó. This is written $X \sim N_2(m, \Sigma)$. A random sample of size n from this distribution (consisting of n 2-D vectors) $\{(x_i, y_i), i = 1 \ldots n\}$ has a sample mean vector $\overline{X}$ and a sample covariance matrix in the mean V where

$$\overline{X} = \begin{pmatrix} \overline{x} \\ \overline{y} \end{pmatrix} = \begin{pmatrix} \frac{1}{n}\sum_{i=1}^{n} x_i \\ \frac{1}{n}\sum_{i=1}^{n} y_i \end{pmatrix}$$

$$V = \begin{pmatrix} u^2(\overline{x}) & u(\overline{x},\overline{y}) \\ u(\overline{y},\overline{x}) & u^2(\overline{y}) \end{pmatrix} = \begin{pmatrix} \frac{1}{n(n-1)}\sum_{i=1}^{n}(x_i-\overline{x})^2 & \frac{1}{n(n-1)}\sum_{i=1}^{n}(x_i-\overline{x})(y_i-\overline{y}) \\ \frac{1}{n(n-1)}\sum_{i=1}^{n}(y_i-\overline{y})(x_i-\overline{x}) & \frac{1}{n(n-1)}\sum_{i=1}^{n}(y_i-\overline{y})^2 \end{pmatrix}$$

Here $u^2(\overline{x})$ and $u^2(\overline{y})$ are squared standard uncertainties in the mean and $u(\overline{x},\overline{y})$ and $u(\overline{y},\overline{x})$ are estimated covariances in the mean. V is a symmetric matrix because the two covariance terms are equal. In a measurement context, the sample size n is a positive integer equal to the number of repeat measurements. The sample mean is itself a random vector which varies from one sample to the next and is normally distributed with population mean vector ì and population covariance matrix $\overline{\Sigma} = \Sigma/n$.

The vector $\overline{X}$ and the matrix k^2V, where k is a coverage factor, define an ellipse with equation

$$(Y-\overline{X})^T(k^2V)^{-1}(Y-\overline{X}) = 1$$

where Y is the position vector of a variable point in the plane. This ellipse encompasses the population mean ì if

$$(m-\overline{X})^T V^{-1}(m-\overline{X}) \le k^2$$

In terms of a statistic U defined by $U = (m-\overline{X})^T V^{-1}(m-\overline{X})$, this condition can be written $U \le k^2$. U is a random variable whose value varies from one sample to the next. The population mean corresponds to a fixed point in the plane and each random sample from the population gives rise to a random ellipse in the plane. The coverage factor k is chosen so that for a given sample, the sample ellipse encompasses the population mean with a 100á% confidence i.e. $pr(U \le k^2) = a$ where $pr()$ denotes the probability that the condition in the brackets holds. If a very large number of samples were taken then 100á% of the corresponding ellipses would encompass the population mean.

The following result (ref. 1) allows the coverage factor associated with a given confidence level (or the confidence level associated with a given coverage factor) to be determined.

If a 2-dimensional vector $\overline{X}$ is normally distributed with mean vector μ and covariance matrix $\overline{\Sigma}$ then the quantity $\tilde{U}$ defined by

$$\tilde{U} = (m-\overline{X})^T \overline{\Sigma}^{-1}(m-\overline{X})$$

has a chi-squared (÷^{2}) distribution with 2 degrees of freedom.

When the effective number of degrees of freedom associated with the measurement is large then the sample covariance matrix in the mean V is a good approximation to $\overline{\Sigma}$ and so U is a good approximation to $\tilde{U}$. Hence U has a ÷^{2} distribution with 2 degrees of freedom.

The confidence level associated with a given coverage factor k is just the probability that U is less than k^2. Thus for a 2-dimensional normal distribution the coverage factor k associated with a given confidence level can be read off from a table of the inverse cumulative distribution function (CDF) of the ÷^{2} distribution with 2 degrees of freedom. In particular, this coverage factor depends only on the confidence level and is independent of the mean vector and the covariance matrix of the normal distribution (i.e. of the location and shape of the distribution).

Example 1: coverage factor for 95% confidence
From statistical tables (e.g. ref. 2), the inverse CDF of the ÷^{2} distribution with 2 degrees of freedom evaluated at 0.95 is 5.99. This implies that $pr(U \le k^2) = 0.95$ when $k^2 = 5.99$. It follows that the coverage factor k is given by $k = 2.45$. Coverage factors for other confidence levels can be calculated in the same way.

Example 2: confidence level for coverage factor *k* = 1.00
From statistical tables (e.g. ref. 2), the CDF of the ÷^{2} distribution with 2 degrees of freedom evaluated at 1.00 is 0.39. This implies that $pr(U \le k^2) = 0.39$ when $k^2 = 1.00$. It follows that the confidence level associated with the coverage factor $k = 1.00$ is 39%. Confidence levels for other coverage factors can be calculated in the same way.

References

[1] K V Mardia, J T Kent and J M Bibby, “Multivariate analysis”, *Academic Press*, 1979.

[2] E. S. Pearson and H. O. Hartley (Eds.), “Biometrika tables for statisticians”, Third Edition, Cambridge University Press