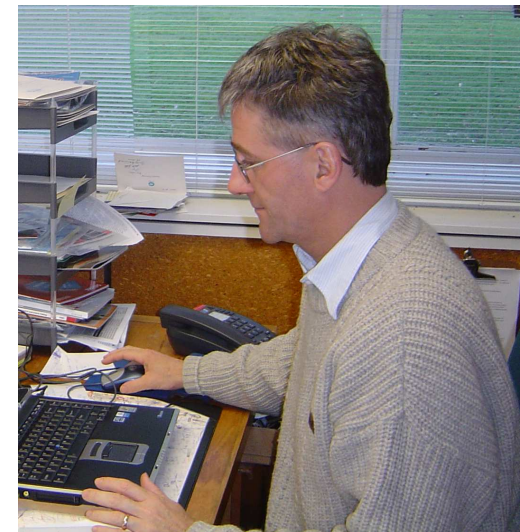# *Uncertainty statements for complex quantities in polar and rectangular coordinates*

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Notes

Overview

Polar representations are popular, but are they a good way of reporting uncertainty in a complex quantity?

■ Uncertainty in the complex plane – geometry

■ Polar → rectangular

■ Rectangular → polar

■ Checks and balances?

■ 'Polarised' uncertainty statements?
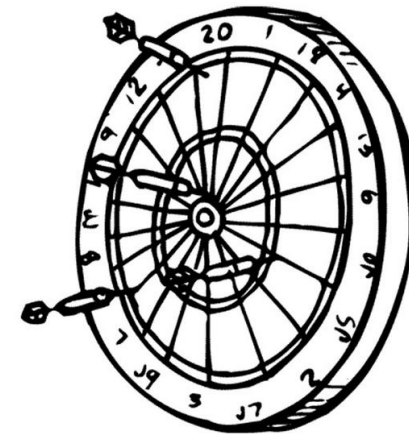
Notes

- This talk is based on some published work (B. D. Hall, *Some considerations related to the evaluation of measurement uncertainty for complex-valued quantities in radio frequency measurements*, Metrologia (2007) **44** L62-L67.)

- Begin by reviewing the geometry of uncertainty in the complex plane: the idea of a region/ellipse (rather than an interval) associated with possible values of the measurand and the relationship of that ellipse to the calculated covariance matrix (i.e., how the geometry of the ellipse is related to the matrix properties)

- Show how to take a statement of uncertainty in polar co-ordinates and calculate a covariance matrix (in rectangular co-ordinates)

- Show how to take a covariance matrix and calculate uncertainty in polar co-ordinates

- Are these calculations safe? Do we loose information about the measurement uncertainty when performing these transformations? How do we find out?

- Can we check to see if the transformation is 'safe'?

- Is there an alternative to polar uncertainty statements?

Uncertainty in the complex quantity

- Uncertainty is associated with a region in the complex plane

- We *think* that the measurand is inside the region

- Units are the same in both directions

Notes

- A complex value is represented as a point in the complex plane.

- Uncertainty in a complex value represents what we know about the likely errors in the real and imaginary components.

- The locus of possible error values forms a region in the complex plane.

- A conventional shape for an uncertainty region is an ellipse centered on the estimate of the quantity.

Covariance matrices

The covariance matrix characterises uncertainty

- ■ Key information:

  - ◆ real cpt: $u(x_{\mathrm{re}})$

  - ◆ imaginary cpt: $u(x_{\mathrm{im}})$

  - ◆ correlation: $r(x_{\mathrm{re}}, x_{\mathrm{im}})$

- ■ Covariance matrix

$$\mathbf{V} = \left[\begin{array}{cc} v_{11} & v_{12} \\ v_{21} & v_{22} \end{array} \right]$$

$$
\begin{aligned}
v_{11} &= u^2(x_{\mathrm{re}}) \\
v_{12} &= u(x_{\mathrm{re}})\, r(x_{\mathrm{re}}, x_{\mathrm{im}})\, u(x_{\mathrm{im}}) \\
v_{21} &= v_{12} \\
v_{22} &= u^2(x_{\mathrm{im}})
\end{aligned}
$$

- ■ The contour of an uncertainty region is defined by

$$(\boldsymbol{\xi} - \boldsymbol{x})' \mathbf{V}^{-1} (\boldsymbol{\xi} - \boldsymbol{x}) = c^2$$
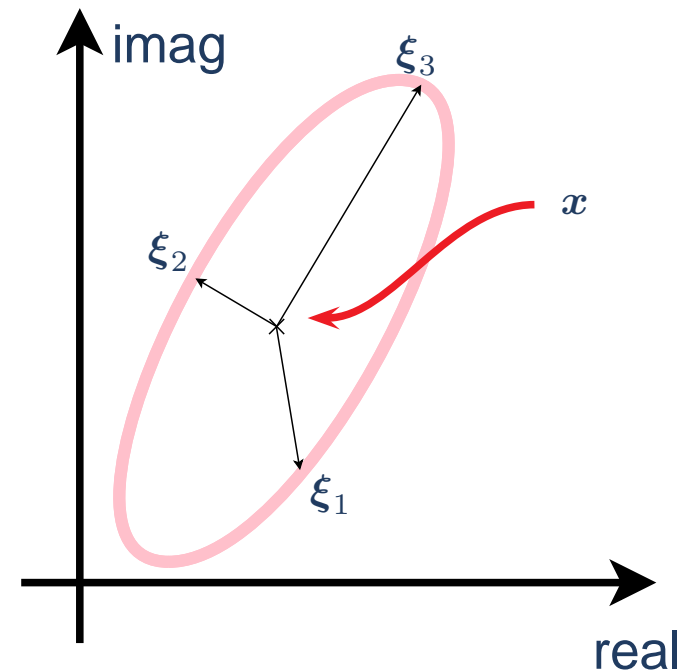
Notes

- In real-valued uncertainty calculations it is common to present uncertainty calculations in terms of a standard deviation associated with the quantity of interest. This is creating a false impression that the standard deviation is the important statistical quantity.

- In fact the important quantities in those calculations are the *variances* and *covariances*.

- In the complex case, the variance-covariance matrix associated with the two components of the complex quantity is fundamental.

- A 2×2 variance-covariance matrix is associated with every influence quantity in order to evaluate the 2×2 variance-covariance associated with the measurement result.

- The variance-covariance matrix determines the shape of the elliptical uncertainty region.

Circular uncertainty regions

A circle is the simplest uncertainty region

- **Real and imaginary uncertainties are equal**

$$u(x_{\mathrm{re}}) = u(x_{\mathrm{im}}) = u(x)$$

- **No correlation:**

$$r(x_{\mathrm{re}}, x_{\mathrm{im}}) = 0$$

- **Simple covariance matrix**

$$\begin{bmatrix} u^2(x) & 0 \\ 0 & u^2(x) \end{bmatrix}$$

imag

$\mathrm{dia} \sim u(\boldsymbol{x})$
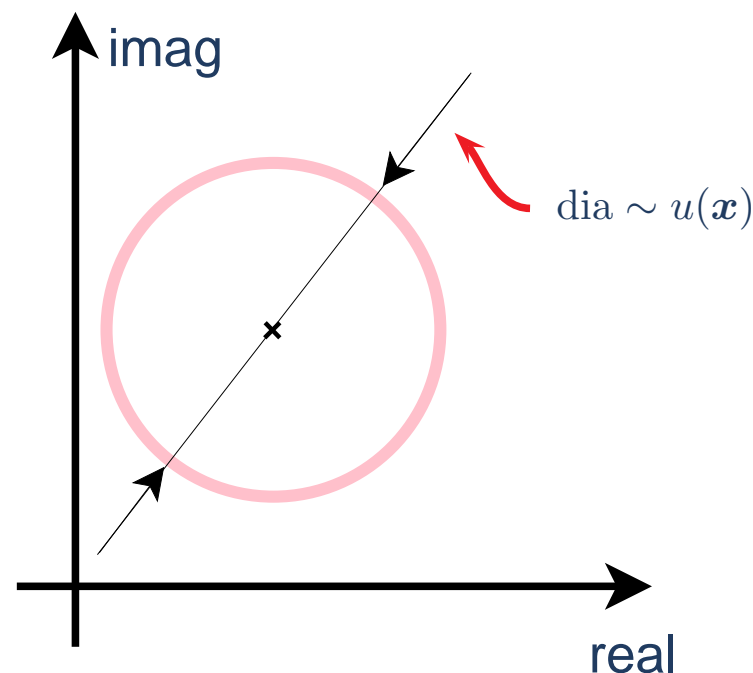
real

Notes

■ A circle in the complex plane is the simplest form of uncertainty region.

■ It represents

◆ errors of equal variance (likelihood of a certain magnitude of error is the same) in the real and imaginary components

◆ independent errors in the real and imaginary components

■ Note, the *distribution* is not circular: a circular uncertainty region does NOT imply a *uniform* circular distribution.

Uncorrelated real and imaginary errors

■ Principal axes are parallel to the real and imaginary axes

■ Diagonal covariance matrix: $\begin{bmatrix} u^2(x_{\mathrm{re}}) & 0 \\ 0 & u^2(x_{\mathrm{im}}) \end{bmatrix}$
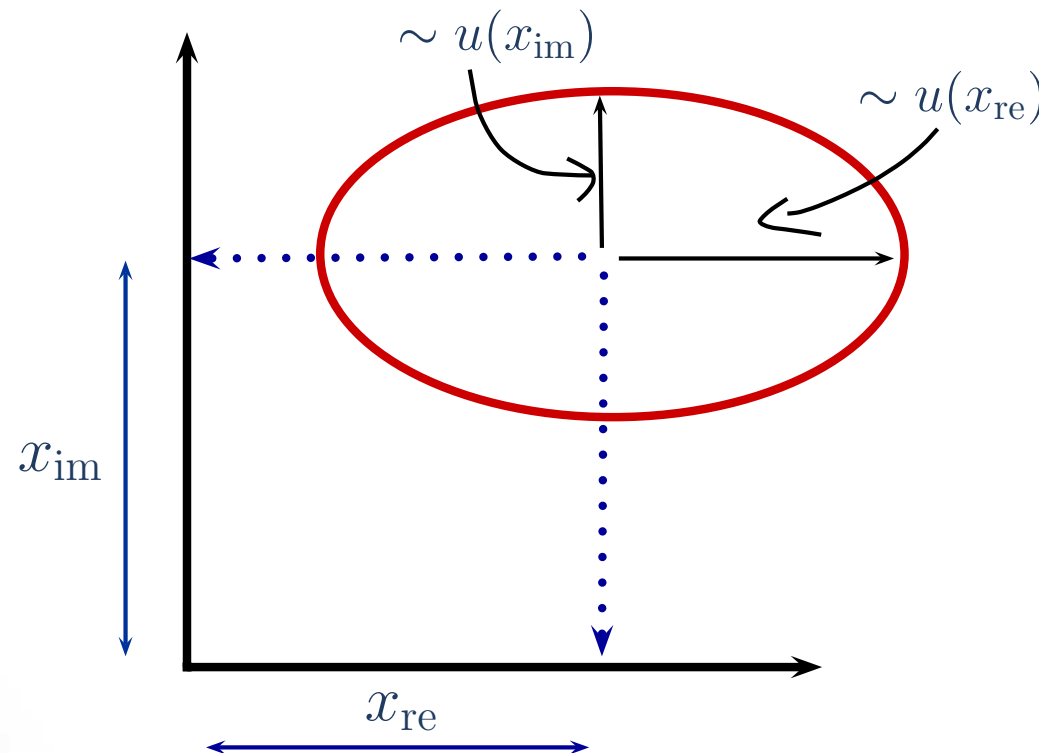
Notes

- If the major and minor axes of the ellipse align with the real and imaginary axes there is no correlation between the estimates of the real and imaginary components.

- The square root of the covariance matrix eigenvalues are proportional to the real and imaginary standard uncertainties

Correlated real and imaginary errors

- Principal axes are rotated

- Covariance matrix: $\begin{bmatrix} u^2(x_{\mathrm{re}}) & r\,u(x_{\mathrm{re}})u(x_{\mathrm{im}}) \\ r\,u(x_{\mathrm{re}})u(x_{\mathrm{im}}) & u^2(x_{\mathrm{im}}) \end{bmatrix}$
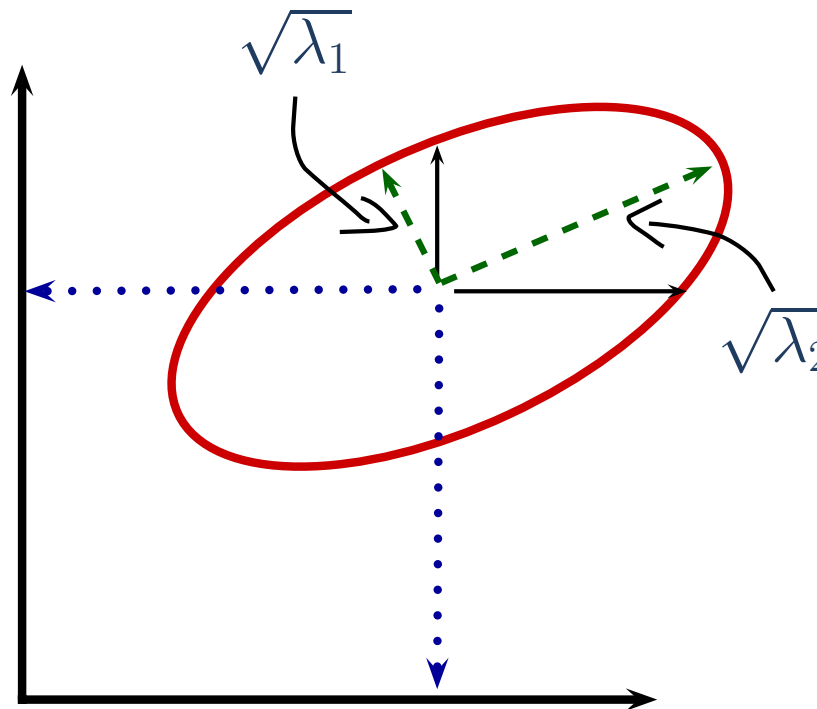
Notes

- The principal axes (along the directions of the covariance matrix eigenvectors) are no longer aligned with the real and imaginary axes.

- This signifies correlation between the estimates of the real and imaginary components

- The covariance matrix eigenvalues do not have a simple relationship to the real and imaginary standard uncertainties

Summary measures of uncertainty

A single summary value for the covariance matrix is useful but two are in common use

■ Total Variance [recommended]

◆ The trace of the covariance matrix

◆ $V_{\mathrm{T}} = \lambda_1 + \lambda_2$

◆ Think of the standard variance of the sum of the real and imaginary components

■ Generalized Variance [not recommended]

◆ The determinant of the covariance matrix

◆ $V_{\mathrm{G}} = \lambda_1 \cdot \lambda_2$ (NB, units are different)

◆ If one component has little or no uncertainty $V_{\mathrm{G}} \to 0$, hiding uncertainty in the other component

Notes

- The variance-covariance has 4 elements, so it is convenient to have single-valued magnitude for the covariance matrix.

- Two common choices are:

    - Total Variance: the trace of the covariance matrix
    - Generalized Variance: the determinant of the covariance matrix

- These can be related to the geometry of the uncertainty region by considering the eigenvalues of the covariance matrix:

    - The Total Variance is equal to the sum of the eigenvalues. It summarises the uncertainty but ignores correlation.

    - The Generalized Variance is equal to the product of the eigenvalues, so it is a measure of the surface extent of the region (proportional to the area squared of the region) – note, it doesn't have the units of variance!

    - Note, these measures do not change under a rotation of the co-ordinate axes (the eigenvalues don't change)

- The Total Variance is preferred for metrology. It is related to the combined standard variance of (the sum of) the real and imaginary components.

- The Generalized Variance is not as useful for metrology because if one component has little or no uncertainty the GV $\rightarrow$ 0, which could hide significant uncertainty in the other component
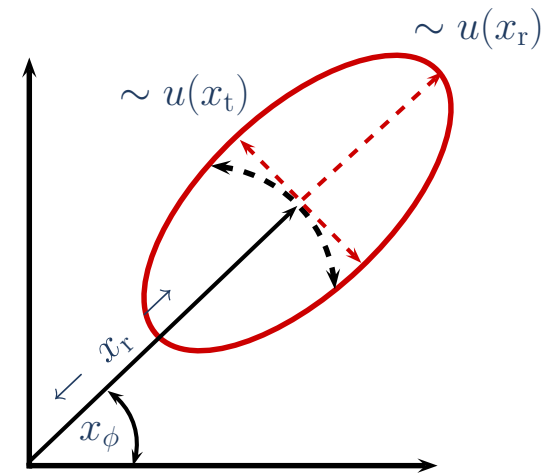
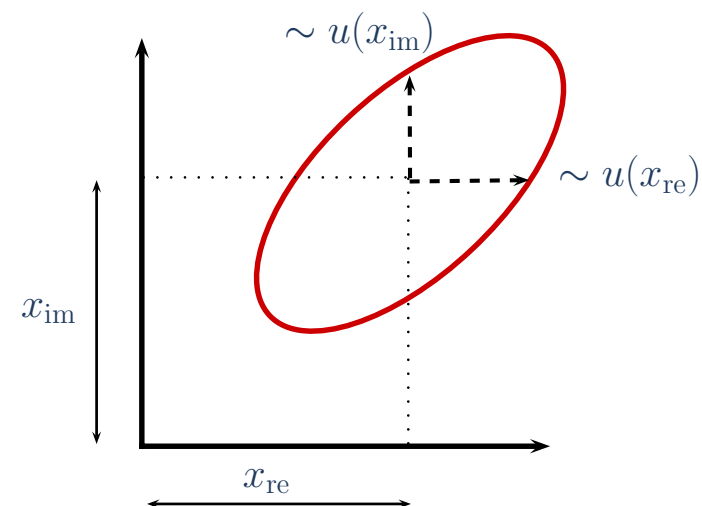# Transforming to rectangular co-ordinates

Procedure (approximate):

- ■ Write down a covariance matrix in coordinates aligned with polar axes

$$\mathbf{V}_{rt} = \begin{bmatrix} u^2(x_r) & 0 \\ 0 & u^2(x_t) \end{bmatrix}$$

$$u(x_t) = x_r \tan u(x_\phi)$$

- ■ Transform to rectangular coordinates by rotation

$$\mathbf{V} = \mathbf{R}_{x_\phi} \, \mathbf{V}_{rt} \, \mathbf{R}'_{x_\phi}$$

Notes

- This method is NOT exact. It may not work very well if the curvature in the polar coordinates is significant.

- The matrix

$$\mathbf{R}_{x_\phi} = \begin{bmatrix} \cos x_\phi & -\sin x_\phi \\ \sin x_\phi & \cos x_\phi \end{bmatrix}$$

- If there is correlation between ϕ and r then the off-diagonal elements of $\mathbf{V}_{\mathrm{rt}}$ need to be entered too.

Transforming from rectangular co-ordinates

Procedure (approximate):

■ Change coordinates to obtain a 'polarised' covariance matrix, aligned with the polar coordinates

$$\mathbf{V}_{\mathrm{rt}} = \mathbf{R}'_{x_\phi} \, \mathbf{V} \, \mathbf{R}_{x_\phi}$$

■ Calculate standard uncertainty in polar coordinates

$$u(x_{\mathrm{r}}) = \sqrt{\mathbf{V}_{\mathrm{rt}}(1,1)}$$

$$u(x_\phi) = \tan^{-1}\left(\sqrt{\mathbf{V}_{\mathrm{rt}}(2,2)}/x_{\mathrm{r}}\right)$$

$$r = \mathbf{V}_{\mathrm{rt}}(1,2)/\sqrt{\mathbf{V}_{\mathrm{rt}}(1,1)\mathbf{V}_{\mathrm{rt}}(2,2)}$$

Notes

- This method is NOT exact. It may not work very well if the curvature in the polar coordinates is significant.

- We do not think that it is a good idea to transform uncertainty statements from rectangular to polar coordinates. However, if you insist, then this is one way to do it.

- In calculating the correlation coefficient r we also assume that the correlation is unchanged by transforming from the 'polarised' coordinates back to polar coordinates. This too is an approximation.

Does it work?

Test by simulation

- Choose μ, e_r and e_ϕ

    1. Simulate observations of x_r and x_ϕ

    2. Apply the procedure to obtain $\mathbf{V}$

    3. Does the uncertainty region contain $\boldsymbol{\mu}$?

    4. Simulate again (go back to step 1)

- Expect close to a 95% success-rate over a large number of simulations

Notes

- Assume that if the procedure works well on data sets similar to those obtained in real measurements, then it is reliable

- Take the standard uncertainties $u(x_\mathrm{r})$ and $u(x_\phi)$ to be the standard deviations of the unknown errors, e_r and e_ϕ

- Use random number generators to simulate experimental results, providing sets of data to test the procedure

- In this case, we will add Gaussian random numbers with zero mean and standard deviations $u(x_\mathrm{r})$ and $u(x_\phi)$ to the polar coordinates of μ to obtain simulated observations

- A general procedure for this could be more complicated (e.g., if finite degrees-of-freedom were considered)

Example

Offset-short (data from METAS calibration report)

■ Data

$$\Gamma_{\mathrm{r}} = 0.995, \ u(\Gamma_{\mathrm{r}}) = 0.013; \qquad \Gamma_{\phi} = 85.34°, \ u(\Gamma_{\phi}) = 0.88°$$

$$\mathbf{V}_{\mathrm{rt}} = \begin{bmatrix} 1.69 \times 10^{-4} & 0 \\ 0 & 2.34 \times 10^{-4} \end{bmatrix}$$

Success-rate in simulations: $94.7 \pm 0.4\%$

■ Increased phase uncertainty

uncertainty	success-rate
$2u(\Gamma_{\phi})$	94 %
$5u(\Gamma_{\phi})$	84 %

Notes

- Assume that the errors in magnitude and phase are independent (so there are no off-diagonal matrix elements)

- The diagonal elements of $\mathbf{V}_{rt}$ are roughly the same size, which means that the ellipse roughly circular

- When the standard uncertainty in phase is increased five-fold, the covariance matrix element increases 25-fold.

- When the standard uncertainty in phase is increased five-fold, it represents a $4°$ standard uncertainty in the phase

Is the intermediate matrix an alternative?

■ The 'polarised' covariance matrix is 'like' an uncertainty statement in polar coordinates

  ◆ radial and tangential uncertainties

■ Homogeneous units are insightful when dealing with a complex quantity (previous example)

  ◆ Why have different tangential units?

  ◆ Perhaps some measurements are not really about the complex quantity?
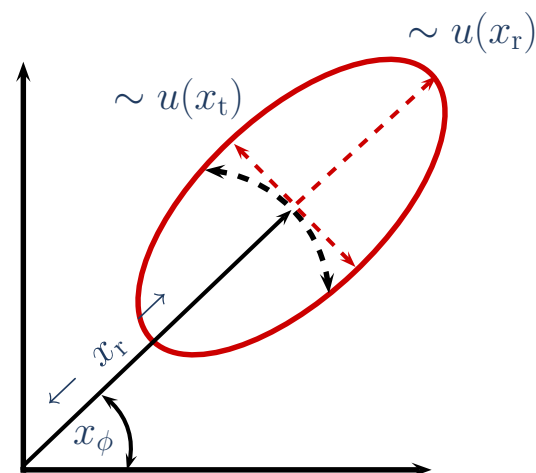
Notes

- The 'polarised' covariance matrix could be reported, instead of the real-imaginary covariance matrix, without any of the loss of information that goes with the transformation to polar coordinates. Would there be any merit in that?

- It is more informative to look at the 'polarised' covariance matrix if one is considering a complex quantity because the units are homogeneous. A statement of uncertainty in polar coordinates on the other hand is hard to visualise in the complex plane.

- For instance, how easy is it to see that the uncertainty region in the offset-short example is nearly circular? Given only the data in polar coordinates its quite hard: given the 'polarised' covariance matrix it is trivial.

- Where does the real 'demand' come from for statements of uncertainty in polar coordinates?

  - Are we actually considering the magnitude and phase as distinct real quantities? (In which case, GUM methods apply and we do not need to get tangled up in bivariate uncertainty propagation.)

  - Are we simply captured by a legacy of engineering formulae that require phase as an argument?

Main points

■ A simple method can transform uncertainties between polar and rectangular coordinates

    ◆ to a covariance matrix from polar uncertainty

    ◆ to polar uncertainty from a covariance matrix

■ The method is approximate

    ◆ Trigonometric functions needed to handle phase

■ The intermediate 'polarised' covariance matrix may be useful in reporting uncertainty without information loss

    ◆ What is the purpose of the measurement?

    ◆ Is the quantity of interest just a component?

Notes

■ The method of transforming between polar and rectangular coordinates is easy to apply

■ The method can loose information in some cases (it is approximate)

■ The intermediate step in which a 'polarised' covariance matrix is obtained is interesting

◆ Could it be used to report results when the polar coordinates have some fundamental meaning to the measurement?

◆ Can it help to wean people off polar coordinates?

■ Why do people really want to work in polar coordinates anyway? (Good to discuss this). A homogeneous set of units is appropriate for a complex quantity? Are we confusing some cases where the problem is univariate with others where it is bivariate?

THANK YOU

The Lower Hutt campus of Industrial Research Ltd

Notes