Mismatch uncertainty:

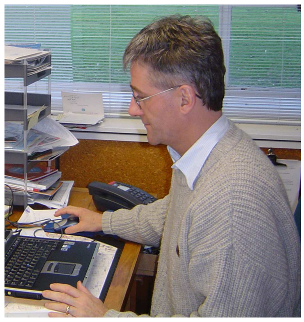
representations for complex calculations

Blair Hall

Measurement Standards Laboratory of New Zealand

Industrial Research Ltd.,
PO Box 31-310, Lower Hutt,
New Zealand.

b.hall@irl.cri.nz



Presented (via internet) to the 29th ANAMET meeting, 6th March, 2008.

Overview

Introduction
Mismatch Uncertainty
– arcsine distribution
Complex uncertainty
– covariance matrix
– circular regions
Unknown phase
– covariances
– product distributions
– three combinations
Propagating uncertainty
Power measurement

- Mismatch Uncertainty
  - What is it?
  - How is it represented in real-valued uncertainty calculations?
- Measurement uncertainty
  - What is the uncertainty of a complex quantity?
  - How is it represented?
- Ignorance about phase
  - When only the magnitude is measured
  - How is that represented?
- Example

This talk is based on recently published work (B. D. Hall, *Metrologia* **44** (2007) L62-L67).

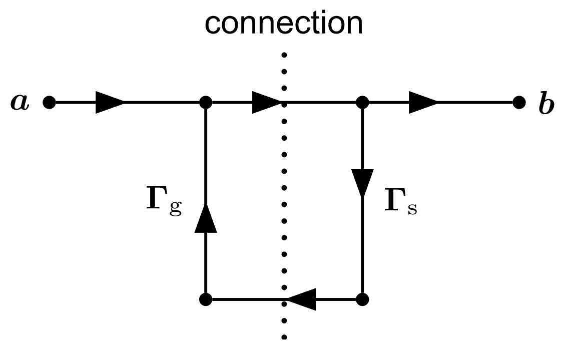
It discusses the evaluation of measurement uncertainty when mismatch is to be considered. Specifically, the talk is concerned with the representation of mismatch uncertainty in problems that involve complex-valued quantities. The related problem, mismatch uncertainty in real-valued uncertainty calculations, is dealt with in many guidelines on uncertainty.

The notion of mismatch is intrinsically linked to the complex (2-dimensional) nature of the measurements. Formal techniques for carrying out complex uncertainty calculations are now available, so a complex representation for mismatch uncertainty is important. This work therefore extends that of Harris and Warner, which lead to the widespread use of the arcsine distribution in real-valued uncertainty calculations (I A Harris and F L Warner, *IEE Proc.* **H-128**, 1981, 35-41).

We have found that when the problem is considered from a two-dimensional perspective there are several simple intuitive representations of mismatch uncertainty. The arcsine distribution is related to the most conservative statement of uncertainty for the problem.

Mismatch Uncertainty

Introduction
Mismatch Uncertainty
– arcsine distribution
Complex uncertainty
– covariance matrix
– circular regions
Unknown phase
– covariances
– product distributions
– three combinations
Propagating uncertainty
Power measurement



- Transmission along a line with imperfect terminations ($\mathbf{\Gamma} \neq 0$)
- The relative phase determines signal amplitude:

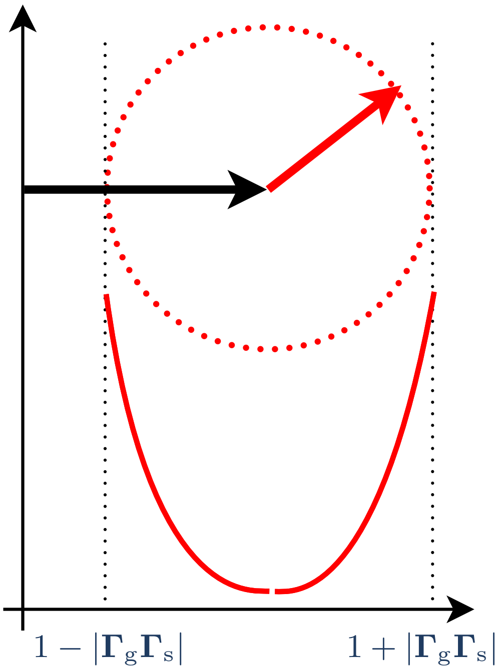
$$\boldsymbol{b} = \frac{\boldsymbol{a}}{1 - \mathbf{\Gamma}_\mathrm{g}\mathbf{\Gamma}_\mathrm{s}}$$

Often, as part of a measurement set-up, non-zero reflection coefficients on each side of a connection plane contribute to uncertainty. Such configurations lead to a series of reflected waves and we cannot predict the incident signal amplitude without full information about the relative phases.

The product $\mathbf{\Gamma}_\mathrm{g}\mathbf{\Gamma}_\mathrm{s}$ is often considered to be an uncertain quantity, because the phase is unknown (perhaps the phases of $\mathbf{\Gamma}_\mathrm{g}$ and $\mathbf{\Gamma}_\mathrm{s}$ are unknown but also there will generally be some unknown phase associated with the connection). Hence we will treat the mismatch uncertainty associated with product $\mathbf{\Gamma}_\mathrm{g}\mathbf{\Gamma}_\mathrm{s}$ as a single influence quantity. (This is important in practice, because otherwise algorithms for uncertainty propagation will probably fail – see: B. D. Hall, *Metrologia* **44** (2007) L62-L67.)

The arcsine distribution

Introduction
Mismatch Uncertainty
– arcsine distribution
Complex uncertainty
– covariance matrix
– circular regions
Unknown phase
– covariances
– product distributions
– three combinations
Propagating uncertainty
Power measurement



- Without 'knowledge' about the relative phase, assume equal likelihood
- The 'arcsine' distribution is the marginal distribution in this case
- The arcsine distribution has a standard uncertainty

$$u(x) = a/\sqrt{2}\,,$$

where $a = |\Gamma_{\mathrm{g}}\Gamma_{\mathrm{s}}|$

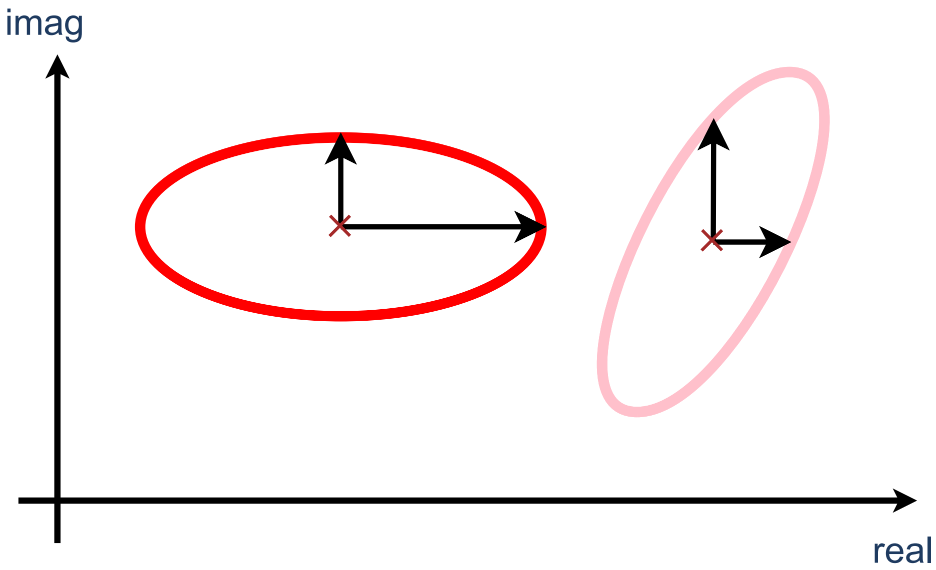
The arcsine distribution has been used to represent mismatch uncertainty in real-valued uncertainty calculations since the work of Harris and Warner (I A Harris and F L Warner, *IEE Proc.* **H-128**, 1981, 35-41).

The idea is that the projection of the distribution of a vector oriented with a uniformly random phase yields a probability density function that is large at its limiting values (asymptotes) and small in the centre of the range.

The arcsine distribution can be used to represent influence quantities if their uncertainty is due to an inherently cyclic phenomenon. For example, a thermal cycle.

Uncertainty in the complex quantity

Introduction
Mismatch Uncertainty
– arcsine distribution
Complex uncertainty
– covariance matrix
– circular regions
Unknown phase
– covariances
– product distributions
– three combinations
Propagating uncertainty
Power measurement



The uncertainty of a complex quantity is associated with a region in the complex plane. An elliptical region is the conventional choice.

A complex quantity has two orthogonal components: the real and imaginary components, in rectangular coordinates, or the magnitude and phase, in polar coordinates. Polar coordinates are problematic for uncertainty calculations, so we prefer to work in rectangular coordinates.

A complex value is represented as a point in the complex plane. Uncertainty in that value as an estimate of the quantity of interest is therefore a region in the plane consisting of points that could reasonably be assigned to the quantity of interest.

A conventional shape for an uncertainty region is an ellipse centered on the estimate of the quantity. If the major and minor axes of the ellipse align with the real and imaginary axes then there is no correlation between the estimates of the real and imaginary components.

Covariance matrix characterises uncertainty

Introduction
Mismatch Uncertainty
– arcsine distribution
Complex uncertainty
– covariance matrix
– circular regions
Unknown phase
– covariances
– product distributions
– three combinations
Propagating uncertainty
Power measurement

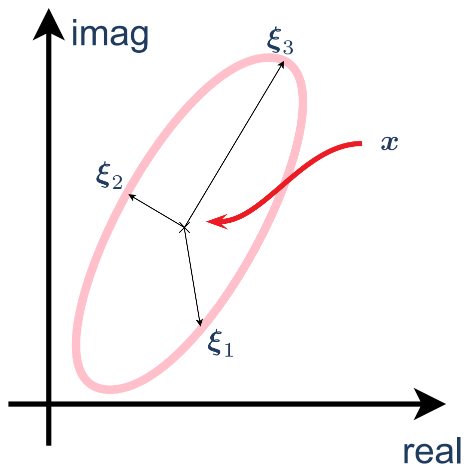
- Three pieces of information:
  - real cpt: $u(x_{\mathrm{re}})$
  - imaginary cpt: $u(x_{\mathrm{im}})$
  - correlation: $r(x_{\mathrm{re}}, x_{\mathrm{im}})$
- The covariance matrix contains all this:

$$\mathbf{V}_{\boldsymbol{x}} = \begin{bmatrix} v_{11} & v_{12} \\ v_{21} & v_{22} \end{bmatrix}$$

$$\begin{aligned} v_{11} &= u^2(x_{\mathrm{re}}) \\ v_{12} &= u(x_{\mathrm{re}})\, r(x_{\mathrm{re}}, x_{\mathrm{im}})\, u(x_{\mathrm{im}}) \\ v_{21} &= v_{12} \\ v_{22} &= u^2(x_{\mathrm{im}}) \end{aligned}$$

- The contour of an uncertainty region is defined by

$$(\boldsymbol{\xi} - \boldsymbol{x})' \mathbf{V}_{\boldsymbol{x}}^{-1} (\boldsymbol{\xi} - \boldsymbol{x}) \le c^2$$



In real-valued uncertainty calculations it is common to evaluate uncertainty as a standard deviation associated with the quantity of interest. In fact the important quantities in those calculations are the variances and covariances.

In the complex problem, the importance of the variances and covariance associated with the two components of the complex quantity is fundamental. A 2×2 variance-covariance matrix needs to be associated with every influence quantity in a problem in order to evaluate the 2×2 variance-covariance associated with the measurement result. The variance-covariance matrix also determines the shape of the conventional uncertainty region.

Circular uncertainty regions

Introduction
Mismatch Uncertainty
– arcsine distribution
Complex uncertainty
– covariance matrix
– circular regions
Unknown phase
– covariances
– product distributions
– three combinations
Propagating uncertainty
Power measurement

The simplest form of uncertainty region is a circle

- Uncertainty in the real and imaginary components is equal

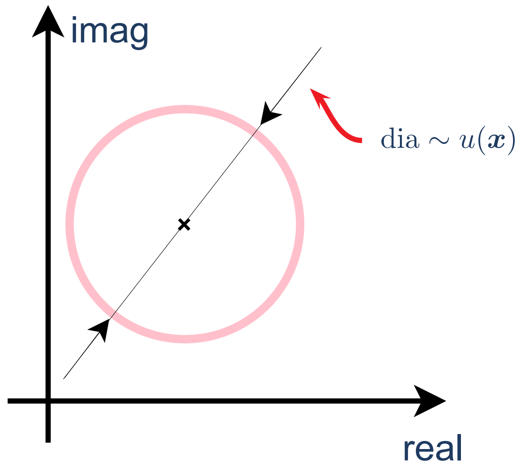
$$u(x_{\mathrm{re}}) = u(x_{\mathrm{im}}) = u(x)$$

- No correlation:

$$r(x_{\mathrm{re}}, x_{\mathrm{im}}) = 0$$

- The covariance matrix is

$$\begin{bmatrix} u^2(x) & 0 \\ 0 & u^2(x) \end{bmatrix}$$



We need to obtain a form of variance-covariance matrix that can represent mismatch uncertainty. First we consider the simplest form of variance-covariance matrix and its representation as an uncertainty region in the complex plane.

This statement of uncertainty essentially says:

- real and imaginary components are independent
- the uncertainty in the real and imaginary components is the same

Note, however, that the simple form of this covariance matrix is not associated with a *uniform* circular distribution. The uncertainty *region* is circular, but the form of distribution is not determined by just the covariance matrix.

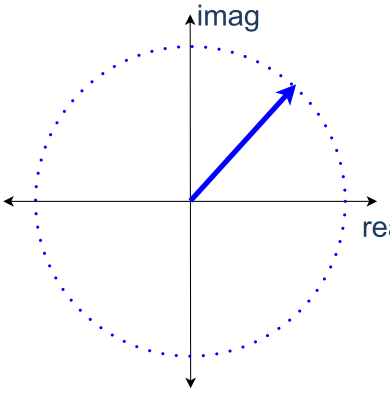
Ignorance about phase

Introduction
Mismatch Uncertainty
– arcsine distribution
Complex uncertainty
– covariance matrix
– circular regions
Unknown phase
– covariances
– product distributions
– three combinations
Propagating uncertainty
Power measurement

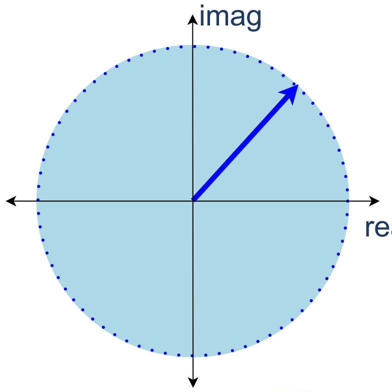
Sometimes, only the magnitude of a signal is measured: the phase is unknown.

- The central estimate of the quantity is 0, the origin!
- The uncertainty depends on what is known about the magnitude

Magnitude is known



Magnitude is bounded above



When measured value is reported as a magnitude in polar coordinates (phase is not measured), the ‘best estimate’ (for the purpose of the uncertainty calculation) may not be a value that could be attributed to the quantity of interest! For example, suppose that we estimate $|\mathbf{\Gamma}| = 0.1$, the locus of points associated with this information is a circle centered on the origin in the complex plane. For the purposes of the uncertainty calculation, the origin should be used as the ‘best estimate’ of the complex quantity (hence, the use of the terminology ‘best estimate’ becomes misleading in this context).

It should also be remembered that the bivariate form of the LPU uses only covariance matrices. There is an underlying assumption that a bivariate gaussian distribution can be associated with the covariance matrix obtained for an estimate of the measurand. The same gaussian assumption is made in the *Guide to the Expression of Uncertainty in Measurement*, for real-valued quantities.

Covariance matrices for these distributions

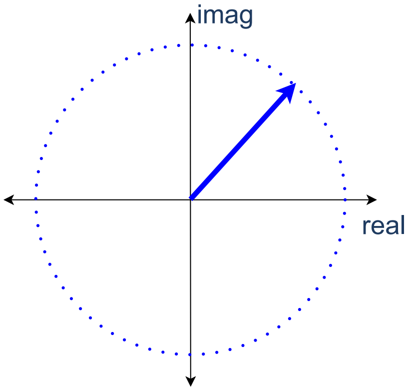
Introduction
Mismatch Uncertainty
– arcsine distribution
Complex uncertainty
– covariance matrix
– circular regions
Unknown phase
– covariances
– product distributions
– three combinations
Propagating uncertainty
Power measurement

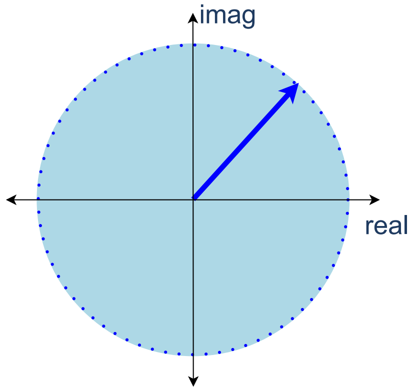
- Magnitude is known so the distribution is a uniform ring of radius a

$$\mathbf{V}_x = \frac{1}{2}\begin{bmatrix} a^2 & 0 \\ 0 & a^2 \end{bmatrix}$$

- Magnitude is bounded so the distribution is a uniform disk of radius a

$$\mathbf{V}_x = \frac{1}{4}\begin{bmatrix} a^2 & 0 \\ 0 & a^2 \end{bmatrix}$$





There is a physical analogy between the simple 2-D probability densities that are associated with type-B uncertainties and the moments of inertia of uniform solid bodies. We are able to use well-known results of classical mechanics for the case of a uniform solid ring and a uniform solid disk, where the respective bodies have unit mass and a radius a. The 2-D moment of inertia tensor for a thin disk, or ring, in the xy plane is equivalent to the variance-covariance matrix of the probability distribution.

Alternatively, noting the radial symmetry of the problem, there is clearly only one covariance matrix term to be calculated. That term is equal to the moment of inertia of a thin disk, or ring, of unit mass and radius a about an axis along a diameter. The result is obtained in many standard mechanics texts.

Note that the uncertainty associated with the disk distribution is *less* than that for the ring distribution.

This appears to be counter-intuitive, because knowing that values inside the ring are not to be considered is presumably informative. However, the ring distribution effectively selects the 'worst-case' magnitudes, giving greater weighting to extreme values and hence greater uncertainty.

The uncertainty of $\Gamma = \Gamma_1\Gamma_2$

Introduction
Mismatch Uncertainty
– arcsine distribution
Complex uncertainty
– covariance matrix
– circular regions
Unknown phase
– covariances
– product distributions
– three combinations
Propagating uncertainty
Power measurement

We are interested in the uncertainty associated with $\mathbf{\Gamma} = \mathbf{\Gamma}_1\mathbf{\Gamma}_2$

- The estimates $\mathbf{\Gamma}_1 = 0$ and $\mathbf{\Gamma}_2 = 0$, so $\mathbf{\Gamma} = 0$.
- The covariance matrices associated with $\mathbf{\Gamma}_1$ and $\mathbf{\Gamma}_2$ have equal diagonal terms and zero off-diagonal terms:

$$\mathbf{V}_{\mathbf{\Gamma}_1} = \begin{bmatrix} \sigma_1^2 & 0 \\ 0 & \sigma_1^2 \end{bmatrix} \quad \text{and} \quad \mathbf{V}_{\mathbf{\Gamma}_2} = \begin{bmatrix} \sigma_2^2 & 0 \\ 0 & \sigma_2^2 \end{bmatrix}$$

- The covariance of the product is

$$\boxed{\mathbf{V}_{\mathbf{\Gamma}} = 2\begin{bmatrix} \sigma_1^2\sigma_2^2 & 0 \\ 0 & \sigma_1^2\sigma_2^2 \end{bmatrix}}$$

Consider two random variables associated with the complex quantities[a]

$$\begin{aligned} \mathbf{\Gamma}_1 &= x_1 + \mathrm{i}y_1 \\ \mathbf{\Gamma}_2 &= x_2 + \mathrm{i}y_2 \ . \end{aligned}$$

The real and imaginary components of these random variables have means of zero. The variances of the real and imaginary components of $\boldsymbol{z}_1$ are both equal to σ_1^2, similarly the variances of the real and imaginary components of $\boldsymbol{z}_2$ are equal to σ_2^2. The covariance between components is zero.

The expectation of the product is zero, i.e.,

$$\boldsymbol{z}_1\boldsymbol{z}_2 = (x_1x_2 - y_1y_2) + \mathrm{i}(x_1y_2 + x_2y_1)$$

and the expectation

$$\begin{aligned} E(x_1x_2 - y_1y_2) &= E(x_1x_2) - E(y_1y_2) \\ &= E(x_1)E(x_2) - E(y_1)E(y_2) \\ &= 0 \end{aligned}$$

similarly $E(x_1y_2 + x_2y_1) = 0$.

[a] The author is grateful to R. Willink for this derivation.

The variances and covariance are then obtained as follows:

$$\begin{aligned} &E\left[(x_1x_2 - y_1y_2)^2\right] \\ &\quad= E\left[x_1^2x_2^2 - 2x_1x_2y_1y_2 + y_1^2y_2^2\right] \\ &\quad= E(x_1^2x_2^2) - 0 + E(y_1^2y_2^2) \\ &\quad= \sigma_1^2\sigma_2^2 + \sigma_1^2\sigma_2^2 \\ &\quad= 2\sigma_1^2\sigma_2^2 \ , \end{aligned}$$

similarly

$$E\left[(x_1y_2 + x_2y_1)^2\right] = 2\sigma_1^2\sigma_2^2$$

and

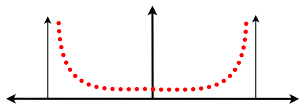
$$\begin{aligned} &E\left[(x_1x_2 - y_1y_2)(x_1y_2 + x_2y_1)\right] \\ &\quad= E(x_1^2x_2y_2) - E(y_2^2x_1y_1) + E(x_2^2x_1y_1) \\ &\qquad - E(y_1^2x_2y_2) \\ &\quad= 0 \end{aligned}$$

Three possible combinations

Introduction
Mismatch Uncertainty
– arcsine distribution
Complex uncertainty
– covariance matrix
– circular regions
Unknown phase
– covariances
– product distributions
– three combinations
Propagating uncertainty
Power measurement

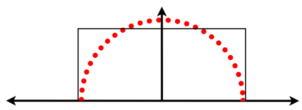
- Known magnitude × known magnitude ⇒ a uniform ring distribution

$$\mathbf{V}_x = \frac{1}{2}\begin{bmatrix} a^2 & 0 \\ 0 & a^2 \end{bmatrix}$$



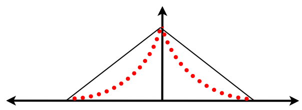
- Known magnitude × unknown magnitude ⇒ a uniform disk distribution

$$\mathbf{V}_x = \frac{1}{4}\begin{bmatrix} a^2 & 0 \\ 0 & a^2 \end{bmatrix}$$



- Unknown magnitude × unknown magnitude ⇒ a non-uniform (linear radial density) distribution

$$\mathbf{V}_x = \frac{1}{8}\begin{bmatrix} a^2 & 0 \\ 0 & a^2 \end{bmatrix}$$



There are three combinations of phase uncertainty that are of interest when considering

$$\mathbf{\Gamma} = \mathbf{\Gamma}_1 \mathbf{\Gamma}_2$$

- $|\mathbf{\Gamma}_1|$ and $|\mathbf{\Gamma}_2|$ are known

  In this case $|\mathbf{\Gamma}|$ is also known, so the ring distribution can be used to describe the uncertainty.

  The standard deviation of the marginal distribution is $a/\sqrt{2}$, i.e. the arcsine distribution.

- one of $|\mathbf{\Gamma}_1|$ and $|\mathbf{\Gamma}_2|$ is bounded and the other is known

  In this case $|\mathbf{\Gamma}|$ is bounded, so the disk distribution can be used to describe the uncertainty.

  Note that the standard deviation of the marginal distribution in this case is $a/2$, which is slightly less than the standard deviation of a uniform distribution $(a/\sqrt{3})$.

  A uniform distribution could be used as a slightly conservative alternative if necessary.

- both $|\mathbf{\Gamma}_1|$ and $|\mathbf{\Gamma}_2|$ are bounded

  In this case the joint density is no longer a uniform distribution in the complex plane (The density is highest at the origin and decreases linearly towards the outer limit).

  The standard deviation of the marginal distribution is $a/\sqrt{8}$, which is slightly less than the standard deviation of a triangular distribution $(a/\sqrt{6})$.

  A triangular distribution could be used as a slightly conservative alternative if necessary.

Propagating uncertainty

Introduction
Mismatch Uncertainty
– arcsine distribution
Complex uncertainty
– covariance matrix
– circular regions
Unknown phase
– covariances
– product distributions
– three combinations
Propagating uncertainty
Power measurement

- The *Law of Propagation of Uncertainty* (LPU) can be applied to the real and imaginary components separately and correlation between these components can also be evaluated.[1]
- The LPU can be expressed in a matrix form for bivariate (complex) problems that obtains the (2×2) covariance matrix[2,3]
- Bivariate LPU can be formulated in terms of sensitivity coefficients (partial derivatives), influence quantity uncertainties and associated correlations using (2×2) matrices[4]

$$\mathbf{V}_{\boldsymbol{y}} = \sum_{i=1}^{m} \sum_{j=1}^{m} \mathbf{U}_{\boldsymbol{x}_i}(\boldsymbol{y})\, \mathbf{R}(\boldsymbol{x}_i, \boldsymbol{x}_j)\, \mathbf{U}'_{\boldsymbol{x}_j}(\boldsymbol{y})$$

[1] *Guide to the Expression of Uncertainty in Measurement*, 1995
[2] K. Weise, *IEEE Trans. Instrum. Meas.*, 1987, **36**, 642-645.
[3] N. M. Ridler and M. J. Salter, *Metrologia*, 2002, **39**, 295-302.
[4] B. D. Hall, *Metrologia*, 2004, **41**, 173-177.

It is not widely recognized that the method described in the *Guide to Expression of Uncertainty in Measurement* (GUM) can be used to evaluate the uncertainty of a complex quantity. The GUM method can treat the real and imaginary components as individual measurands and evaluate their standard uncertainties, as well as the covariance associated with the two components. This information can be used to construct the covariance matrix.

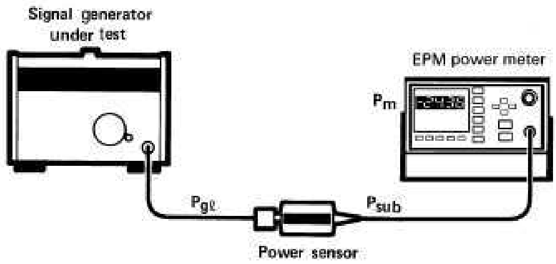
The LPU for real-valued quantities described in the GUM is only a particular case of a general multivariate procedure that has been outlined by Weise. Ridler and Salter presented the bivariate form of this and applied it to some RF measurement scenarios. Hall has described an alternative formulation of the method that may be easier to apply. All three approaches are mathematically equivalent.

Unlike the presentation in the GUM, the bivariate (and multivariate) form of the LPU has been described using matrix notation. However, it is possible to define (2×2) matrix quantities, in the bivariate case, that assume roles analogous to scalar quantities in the GUM's LPU. In this way, it is easier to identify steps in the data processing with actual quantities of interest in the experiment. It is also possible to perform much of the analysis using complex calculus, rather than working with the real and imaginary components of each quantity.

Example: power measurement

Introduction
Mismatch Uncertainty
– arcsine distribution
Complex uncertainty
– covariance matrix
– circular regions
Unknown phase
– covariances
– product distributions
– three combinations
Propagating uncertainty
Power measurement

Simple power measurement scenario:[5][6]



- measurement at 2.4 GHz (Bluetooth and IEEE 802.11b wireless LAN radio systems)
- Agilent E4433A signal generator
- Agilent E4418B power meter and 8481A power sensor

[5]Numerical data from *Fundamentals of RF and Microwave Power Measurement*, (Agilent Technologies Inc., AN 1449-3, USA, 2003).

[6]Excel worksheet available at `http://mst.irl.cri.nz`

We have used the data available in the Agilent Technical Note to evaluate the measurement uncertainty of a realistic power measurement using the ideas just presented.

The calculation turned out to be an interesting exercise. The details are explained in detail in the downloadable example from `mst.irl.cri.nz`.

Here we present just a summary of the main results.

– ring distributions

Introduction
Mismatch Uncertainty
– arcsine distribution
Complex uncertainty
– covariance matrix
– circular regions
Unknown phase
– covariances
– product distributions
– three combinations
Propagating uncertainty
Power measurement

Mismatch uncertainties (during the measurement and during meter calibration are associated with ring distributions) can be treated as arcsine distributions (this is done in the Note)

- The relative combined standard uncertainty is 1.95%.
- The mismatch uncertainty, during the measurement, is dominant
- The most important contributions to the uncertainty are:

quantity x	relative size $u(x)/u_c$	contribution (%) $u^2(x)/u_c^2$
Mismatch	0.72	52.4
Calibration factor	0.44	18.9
Meter gain	0.51	26.5
Instrument noise	0.15	2.2

The ‘Meter Gain’ referred to is a proportionality constant between the meter reading and the ‘substituted power’, determined during meter ‘calibration’ with a known source. The ‘instrument noise’ is a multiplicative noise gain term associated with each reading.

The ‘Meter Gain’ is fixed during calibration, so it is a systematic error in subsequent measurements. The ‘substituted power’ is a random error, which is independent for each reading. Calibration factor is also a systematic error.

What we see here is that mismatch is the dominant uncertainty.

– disk distributions

Introduction
Mismatch Uncertainty
– arcsine distribution
Complex uncertainty
– covariance matrix
– circular regions
Unknown phase
– covariances
– product distributions
– three combinations
Propagating uncertainty
Power measurement

The reflection coefficients of the internal (calibration) power source and the generator are bounded above. So disk distributions are appropriate representations of the uncertainty

- The relative combined standard uncertainty is reduced to 1.66%.
- The most important contributions to the uncertainty are:

quantity x	relative size $u(x)/u_c$	contribution (%) $u^2(x)/u_c^2$
Mismatch	0.60	36.2
Calibration factor	0.51	26.2
Meter gain	0.59	34.6
Instrument noise	0.17	3.2

An inferior uncertainty statement has improved the accuracy!

By using disk distributions instead of rings (arcsine), the uncertainty is reduced. Mismatch no longer dominates. It now has equal importance with the 'meter gain' set during meter 'calibration'.

Although the data do not suggest that a disk distribution could be associated with the sensor reflection coefficient, it is interesting to consider how the results would change if that were done. The table shows that mismatch now comes third, after the calibration factor uncertainty and the meter gain uncertainty. The relative combined standard uncertainty is now 1.49%.

quantity x	relative size $u(x)/u_c$	contribution (%) $u^2(x)/u_c^2$
Mismatch	0.47	22.4
Calibration factor	0.57	32.4
Meter gain	0.64	41.5
Instrument noise	0.19	3.7

It is also interesting to note that even if the full complex sensor reflection coefficient were measured, without a corresponding complex measurement of the calibration source the information about the product still lacks phase information. So the uncertainty of the product will still be a ring or disk.

Final remarks

Introduction
Mismatch Uncertainty
– arcsine distribution
Complex uncertainty
– covariance matrix
– circular regions
Unknown phase
– covariances
– product distributions
– three combinations
Propagating uncertainty
Power measurement

- 2-D representation is better than 1-D
- Unknown phase ⇒ radially symmetric distribution at the origin

- Unknown phase in one factor ⇒ unknown phase in product
- Distributions have simple diagonal covariance matrices
- Three distributions for mismatch uncertainty
  - Ring; disk; r-density
  - Ring distribution (arcsine) is most conservative
  - Is mismatch uncertainty being over-stated?